Fatigue safety factor general formula proposition for the prestressed components subjected to arbitrary CA stress cycling process

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Abstract: *Known Goodman concept of determining the fatigue safety factor is derogated in this paper. It is demonstrated that, in the presence of static prestress, the fatigue strength amplitude calculated after Goodman's criterion is less than the real one. The mistake insight is on the "safe side" in fatigue design computations, but still wrong. This imperfection had been perceived in some books and other papers, but the unique, general formula for determining the fatigue safety factor in the presence of static prestress has not been offered. In this paper, the unique formula for determining the fatigue strength amplitude and fatigue safety factor of components subjected to constant amplitude (CA) stress cycling process in the presence of static prestress is derived. The extremely simple way for determining the HCF life of the prestressed components subjected to an arbitrary CA stress cycling process for known S-N curve and mean and amplitude stresses, on the basis of Goodman line in Haigh diagram, is suggested, as well.*

Key words: Haigh diagram, mean stress, static prestress, load line, fatigue strength amplitude.

1. Introduction

In nowdays of strongly developed probabilistic approaches to fatigue assessments, deterministic approach is however still in use and therefore still important, especially in the design phase of the machine parts, structural components and joints. Such components are frequently subjected to the high cycle fatigue (HCF) stress cycling process and therefore the stress approach to fatigue design is suitable. This approach has been more than hundred years based on the concept of Goodman (straight) line [1] in Haigh diagram and corresponding fatigue safety factor. Goodman line is based on a huge number of testings and it is more or less unquestionable and generaly accepted in design community. It is a locus of fatigue fracture states, i. e. a locus of limit values of amplitude stresses for the certain mean stress or conversely. It connects the end points (0, σ_{-1}) and (σ_F, 0) of Haigh diagram, where σ_{-1} is the endurance limit of observed component at stress ratio $r = \sigma_{min}/\sigma_{max} = -1$ and σ_F is some static property of material strength (originaly by Goodman [1] – ultimate strength), see Fig. 1. Consequently, the safety factor for any point of this line equals unity. For any mean stress x and corresponding amplitude stress y, its equation is

$$y = \frac{\sigma_{-1}}{\sigma_F}\left(\sigma_F - x\right) \ . \qquad (1)$$

A problem arises when determining a point of Goodman line which gives a correct limit value of the fatigue strength amplitude σ_A. After Goodman, this problem is bypassed: one has not to determine the fatigue strength amplitude in order to get the safety factor. The result is bad: fatigue safety factor obtained is sometimes correct, and sometimes incorrect! The mentioned imperfection had been perceived and the concept of the load line has been introduced in fatigue calculations, e.g. [2, 3, 4, 5, 6]. After this concepts, the fatigue strength, it is the limiting value of the amplitude stress, is determined with intersection point of the load line and the Goodman line. Following these concepts, one has to think about how to chart a load line to get the fatigue strength. The readers are instructed to derive the fatigue strength themselves. So, these basicaly correct approaches didn't result with the an unique anlytical expression for determining the correct value of the fatigue strength. The reason is simple: the authors didn't perceive that the resulting stress history of some component is not only a simple sum of the mean and alternating stresses, but it is always a sum of static prestress and the source constant amplitude (CA) stress cycling process which generally has its own mean stress! The load increase doesn't affect the static prestress. It makes only the amplitude and the mean stresses of the source stress cycling process to increase along the path of the load line which origin is therefore moved along the abscissa of the Haigh diagram for the value of the static prestress, see Fig. 2. For a particular source stress cycling process the load line is a single one. In a word, all imperfections in determining the fatigue safety factor rises from no distinguishing among a mean stress and a static prestress.

All of that had been perceived and the correct expression for the safety factor in the case of prestressed bolt had been offered e.g. by Shigley and Mischke [7], and correct general expression for determination the fatigue strength in the case of static prestressing had been obtained by author [8, 9, 10], but there were not enough reverberation in professional ambiences. That is a reason for this paper.

In the next sections the correct expressions for a fatigue safety factors are derived, compared with that after Goodman, and discussed.

2. Fatigue strength and safety factor in the absence of static prestress

In a simple case where the machine or structural (unnotched) component, joint or specimen are subjected to the CA stress cycling process with mean stres σ_m and amplitude stress σ_a, without static prestressing, the load straight line is determined with origin point (0, 0) of Haigh diagram and its slope σ_a/σ_m, see Fig. 1. The stressess σ_a and σ_m vary along the load line (which is also a line of constant stress ratio r). Its equation is

$$y = \frac{\sigma_a}{\sigma_m} x \ . \qquad (2)$$

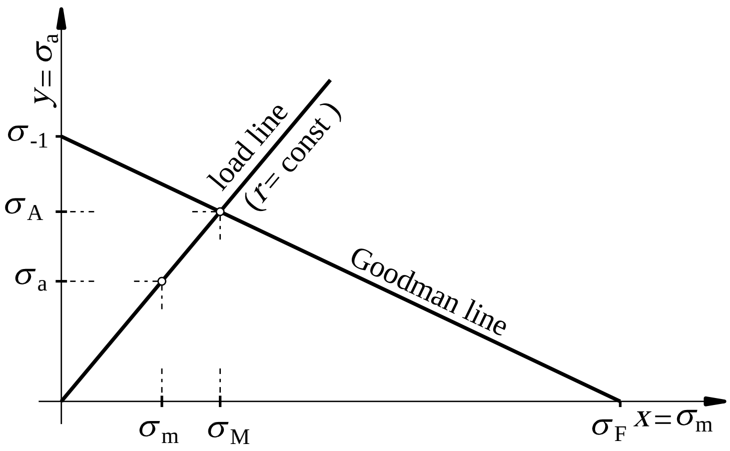

Fig. 1: Determining the fatigue strength when no static prestress is present

The limiting values σ_A and σ_M of σ_a and σ_m are the amplitude and mean stress of the fatigue strength respectively, which are determined with intersection point of the load line (2) and the Goodman line (1). Solving Eqs. (1) and (2) yields

$$y = \sigma_A = \frac{1}{\dfrac{1}{\sigma_{-1}} + \dfrac{\sigma_m}{\sigma_a \sigma_F}} = \frac{1}{\dfrac{1}{\sigma_{-1}} + \dfrac{1+r}{1-r}\dfrac{1}{\sigma_F}} \tag{3}$$

$$x = \sigma_M = \frac{1}{\dfrac{\sigma_a}{\sigma_m \sigma_{-1}} + \dfrac{1}{\sigma_F}} = \frac{1}{\dfrac{1-r}{1+r}\dfrac{1}{\sigma_{-1}} + \dfrac{1}{\sigma_F}} \; . \tag{4}$$

where $r = \sigma_{min} / \sigma_{max}$ is a stress ratio (or a stress cycle assymetry factor). Obviously, σ_A and σ_M do not depend on σ_a and σ_m, but do depend on their ratio, it is on the stress ratio r.

The fatigue safety factor s_f here is equal to the ratio of σ_A and σ_a, but also, only in this simple case, to the ratio of $\sigma_A + \sigma_M$ and $\sigma_{max} = \sigma_m + \sigma_a$ and to the ratio of σ_M and σ_m

$$s_f = \frac{\sigma_A}{\sigma_a} = \frac{\sigma_A + \sigma_M}{\sigma_m + \sigma_a} = \frac{\sigma_r}{\sigma_{max}} = \frac{\sigma_M}{\sigma_m} = \frac{1}{\dfrac{\sigma_a}{\sigma_{-1}} + \dfrac{\sigma_m}{\sigma_F}} \tag{5}$$

where σ_r is the fatigue strength at r stress ratio, expressed in a term of maximum stress.

Obviously, the fatigue safety factor equals the Goodman safety factor, which means that Goodman's formula is valid for such a case of stressing.

3. Fatigue strength and safety factor in the case of statical prestressing

Above presented Goodman fatigue safety factor is generaly accepted for any mean stress regardless of its nature, and here is hidden a mistake. Namely, when component is staticaly prestressed and after that subjected to the source (working) stress cycling process of the certain stress ratio *r*, the static prestress doesn't participate in load and stress increasing. Thus, static prestress σ_{pr} stays same and only the working mean stress σ_m and the amplitude stress σ_a of the source stress cycling process increase, of course, along the load line path which therefore has the same slope as the source stress cycling process has. So, the origin of the load line is moved for the value of mean stress along the abscissa (in Haigh diagram), Fig. 2a, and in the point (σ_{pr}, σ_{pr}) along the symetrale of Smith diagram, Fig. 2b. As Smith diagram has a possibility to present the stress states also in a time-stress diagram, it is more appropriate in order to demonstrate the states of all stresses and its changes.

Obviously, the load line in Haigh diagram passes the point (σ_{pr}, 0) at the slope σ_a/σ_m. It is defined by equation

$$y = \frac{\sigma_a}{\sigma_m}\left(x - \sigma_{pr}\right). \tag{6}$$

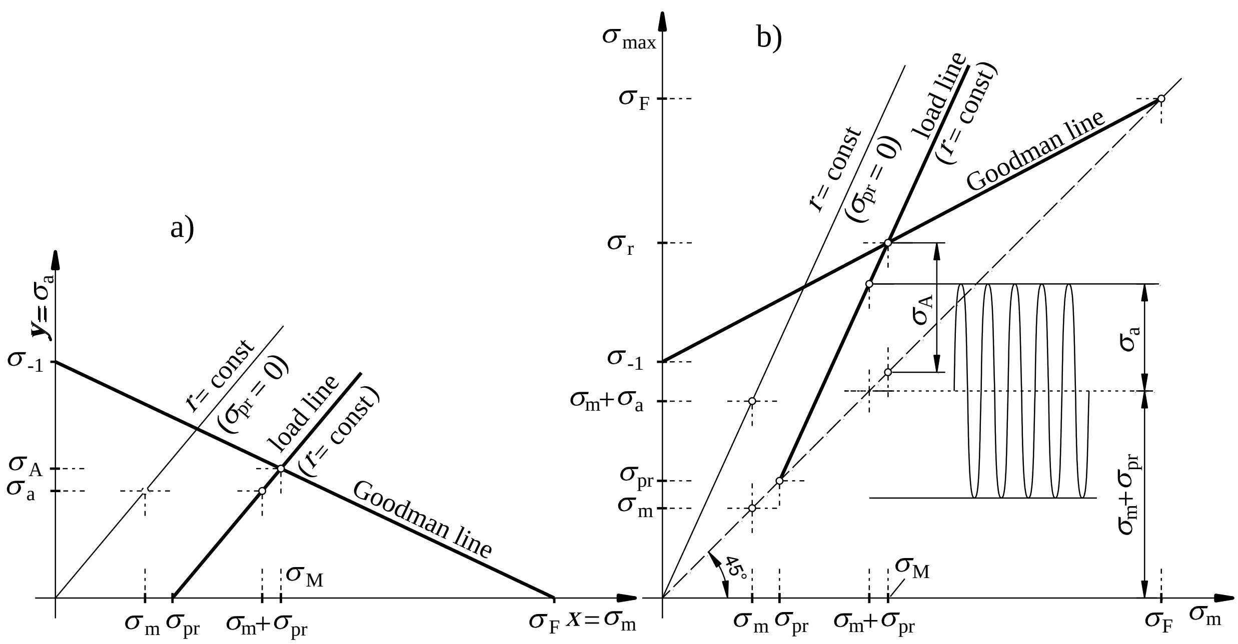

Fig. 2: Determining the fatigue strength in the presence of static prestress
a) in Haigh diagram b) in Smith diagram

The values of the σ_a and σ_m increase along the load line and its limiting values σ_A and σ_M, it is the fatigue strength, are determined (in Haigh diagram) with intersection point of the load line (6) and the Goodman line (1). So, solving Eqs. (1) and (6) yields:

$$\sigma_{\mathrm{A}} = \frac{1-\sigma_{\mathrm{pr}}/\sigma_{\mathrm{F}}}{\dfrac{1}{\sigma_{-1}}+\dfrac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{a}}\sigma_{\mathrm{F}}}} = \frac{1-\sigma_{\mathrm{pr}}/\sigma_{\mathrm{F}}}{\dfrac{1}{\sigma_{-1}}+\dfrac{1+r}{1-r}\dfrac{1}{\sigma_{\mathrm{F}}}} \quad . \tag{7}$$

The fatigue safety factor is the ratio of limiting value σ_{A} of stress amplitude σ_{a} and the stress amplitude itself. It is obtained:

$$s_{\mathrm{f}} = \frac{\sigma_{\mathrm{A}}}{\sigma_{\mathrm{a}}} = \frac{1-\sigma_{\mathrm{pr}}/\sigma_{\mathrm{F}}}{\dfrac{\sigma_{\mathrm{a}}}{\sigma_{-1}}+\dfrac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{F}}}} \quad . \tag{8}$$

After Goodman, the limiting values $\sigma_{\mathrm{A,G}}$ and $\sigma_{\mathrm{M,G}}$ of σ_{a} and σ_{m}, it is the fatigue amplitude limit, is determined, as above, with intersection point of the Goodman line (1) and the straight line passing the origin of the Haigh diagram and the point ($\sigma_{\mathrm{pr}} + \sigma_{\mathrm{m}}$, σ_{a}), see Fig. 3. In that figure, the latter straight line has been named as Goodman load line because it corresponds to Goodman safety factor. But, it cannot be a load line, because it is not the path along which the stresses increase! The static prestress cannot participate in load and stress increase, it stays same! Anyhow, in accordance with Goodman, it is obtained:

$$\sigma_{\mathrm{A,G}} = \frac{\sigma_{\mathrm{a}}}{\dfrac{\sigma_{\mathrm{a}}}{\sigma_{-1}}+\dfrac{\sigma_{\mathrm{pr}}+\sigma_{\mathrm{m}}}{\sigma_{\mathrm{F}}}} \qquad \sigma_{\mathrm{M,G}} = \frac{\sigma_{-1}}{\dfrac{\sigma_{\mathrm{a}}}{\sigma_{\mathrm{pr}}+\sigma_{\mathrm{m}}}+\dfrac{\sigma_{-1}}{\sigma_{\mathrm{F}}}} \quad . \tag{9}$$

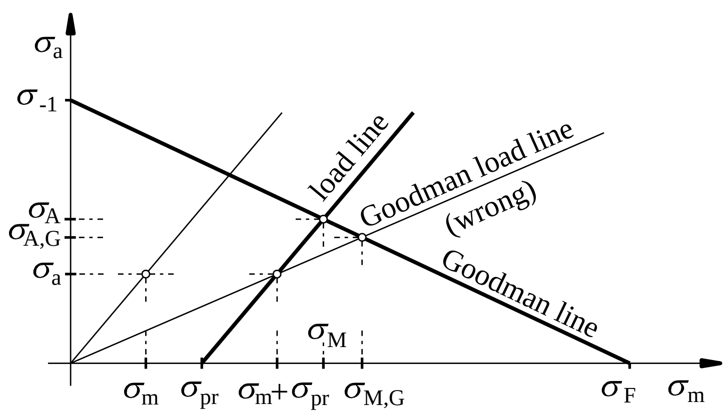

Fig. 3: Demonstration of wrong determination of fatigue strength after Goodman for staticaly prestressed components

Consequently, the known fatigue safety factor after Goodman is obtained once again:

$$s_{\mathrm{f,G}} = \frac{\sigma_{\mathrm{A,G}}}{\sigma_{\mathrm{a}}} = \frac{1}{\dfrac{\sigma_{\mathrm{a}}}{\sigma_{-1}}+\dfrac{\sigma_{\mathrm{pr}}+\sigma_{\mathrm{m}}}{\sigma_{\mathrm{F}}}} \quad . \tag{10}$$

Obviously, in the presence of the static prestress, the fatigue strength amplitude $\sigma_{A,G}$ after Goodman is always less than the real one σ_A, for the same ratio as Goodman fatigue safety factors is less than the real one.

An extremely great mistake arises when the source stress process is of $r = -1$ stress ratio, Fig. 4. In such a case, the real fatigue strength amplitude is determined again by the intersection point of the Goodman line (1) and the load line $\sigma_m = \sigma_{pr}$. It is obtained:

$$\sigma_A = \frac{\sigma_{-1}}{\sigma_F}\left(\sigma_F - \sigma_{pr}\right) = \sigma_{-1}\left(1 - \sigma_{pr}/\sigma_F\right) \ . \tag{11}$$

The real fatigue safety factor is:

$$s_f == \frac{\sigma_{-1}}{\sigma_a}\left(1 - \sigma_{pr}/\sigma_F\right) \ . \tag{12}$$

After Goodman, the fatigue safety factor is then

$$s_{f,G} = \frac{1}{\dfrac{\sigma_a}{\sigma_{-1}} + \dfrac{\sigma_{pr}}{\sigma_F}} = \frac{\sigma_{-1}}{\sigma_a}\frac{1}{1 + \dfrac{\sigma_{pr}\sigma_{-1}}{\sigma_F\sigma_a}} \ . \tag{13}$$

The ratio $s_f / s_{f,G} = \sigma_A / \sigma_{A,G}$ becomes

$$\frac{s_f}{s_{f,G}} = \left(1 - \frac{\sigma_{pr}}{\sigma_F}\right)\left(1 + \frac{\sigma_{pr}}{\sigma_F}\frac{\sigma_{-1}}{\sigma_a}\right) \tag{14}$$

and obviously, it is much greater than one.

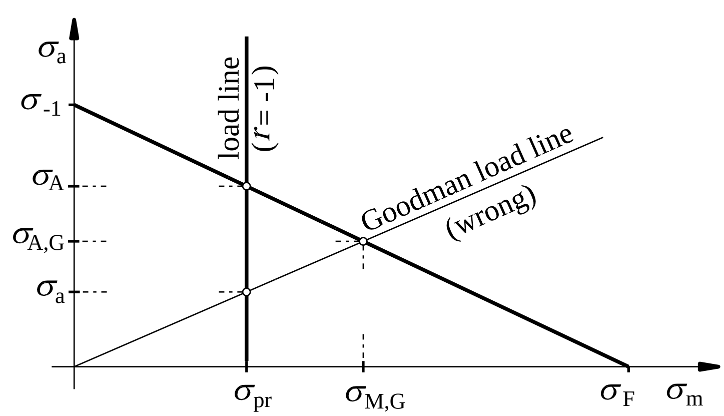

Fig. 4: Comparison of Goodman's and a real fatigue strength amplitude at the presence of static prestress and at $r = -1$ stress ratio of the source stress cycling

It is appropriate to notice that there is a limiting value $\sigma_{pr,b}$ of static prestress σ_{pr} for which the fatigue strength amplitude σ_A equals the stress amplitude σ_a, Fig. 5. For any $\sigma_{pr} > \sigma_{pr,b}$, σ_A becomes less than σ_a and fatigue safety factor less than one. It is not difficult to obtain $\sigma_{pr,b}$:

$$\sigma_{pr,b} = \sigma_F \left(1 - \sigma_a / \sigma_{-1}\right) - \sigma_m \ . \tag{15}$$

Thus, the ratio $\sigma_{pr,b}/\sigma_{pr}$ could be also taken as fatigue safety factor, especcialy in some special, very rear cases, when amplitude stress stays constant by increasing the load.

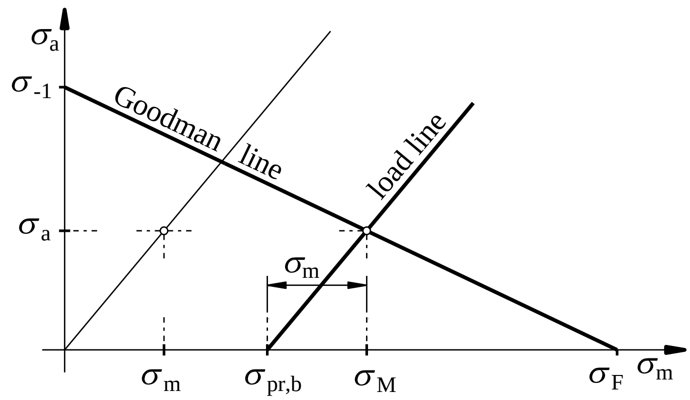

Fig. 5: Limiting value of static prestress

The correct formulae (7) and (8), just like (3) and (5), for determining the fatigue amplitude strength and fatigue safety factor can be applied also to the finit life of components. It is necessary only to change there the endurance limit σ_{-1} with finite life fatigue strength σ_{-1N} where latter is determined after Woehler (or Basquin):

$$\sigma_{-1,N} = \sigma_{-1} \left(N_\infty / N_f \right)^m \tag{16}$$

where N_f is the fatigue life expressed in the number of cycles and N_∞ and m are the fatigue life at the knee and the slope of the Woehler curve, respectively.

4. Fatigue life estimation

The problematics dealth with above is related to HCF and could be applied in a low cycle fatigue (LCF) if dealing with true stresses. In both cases it doesn't impact a fatigue life assessments, because all the stresses are then situated on the Goodman line and one has not to take into account a load line, it is has not to distinguish among mean stress and static prestress. Thus, a Manson, Morrow and other formulae for total strain, basicaly derived from Goodman line stay same also in the case of static prestressing and can be used for determining the fatigue life in the zone of LCF.

However, in the zone of HCF, the fatigue life can be determined in an extremely simple way. Namely, for a given amplitude σ_a and mean σ_m stresses, which lies on Goodman line because they are limiting stresses in the same time, regardless the mean stress comprehends the static prestress or not, the Goodman line in Haigh diagram is determined with points (σ_m, σ_a) and (σ_F, 0), Fig. 7. Its equation is:

$$y = \frac{\sigma_a}{\sigma_m - \sigma_F}\left(x - \sigma_F\right). \tag{17}$$

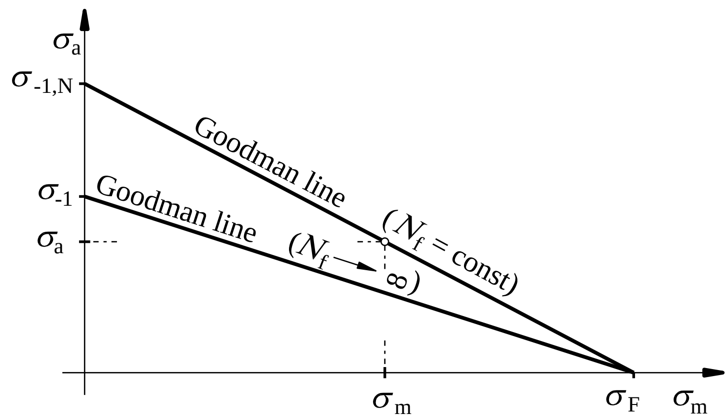

Fig. 7: Determining finit life Goodman line for the certain σ_a, σ_m and σ_F

The finite life fatigue strength σ_{-1N} represents the value of ordinate for zero abscissa:

$$\sigma_{-1,N} = \frac{\sigma_a \sigma_F}{\sigma_F - \sigma_m} = \frac{\sigma_a}{1 - \sigma_m / \sigma_F}. \tag{18}$$

The fatigue life is now obtained from the Woehler curve equation:

$$N_f = N_\infty \left(\frac{\sigma_{-1}}{\sigma_{-1N}}\right)^m = N_\infty \left[\frac{\sigma_{-1}}{\sigma_a}\left(1 - \sigma_m / \sigma_F\right)\right]^m. \tag{19}$$

If dealing in LCF zone, similar procedure, but with true stresses, can also be applied for the estimation of the fatigue life. Since LCF zone is also a zone of elastic-plastic strains, the attention must then be paid to determining the fatigue stress concentration factor which differs from that in HCF zone which is mostly a zone of only elastic strain .

5. Concluding remarks

From the reading-piece presented herein, the following concluding remarks can be derived:

- It is explained that well-known Goodman formula for determining the fatigue safety factor for certain CA stress cycling process is not correct if the component is previously subjected to static prestressing. The correct formula is derived and its use is suggested.
- The mentioned mistake doesn't affect the methods of LCF fatigue assessments (Morrow, Manson, Berkovitz) based on Goodman line in Haigh diagram.
- The limiting value of static prestress is derived which results with critical fatigue safety factor.

- The simple formula for estimating the fatigue life of a component for the certain CA stress cycling process and the certain static prestress is derived and suggested.
- Application of the formulae obtained leads to more robust design, or – for the same design – to increasing the acurracy of assessments.

As presented, Goodman's imperfection in determining the fatigue safety factor in the presence of static prestress, arised from equalizing the mean stress and static prestress, is "on the safe side". That is why its application in fatigue assessments of structures, components and joints couldn't make any harm and that is why this imperfection has not been yet derrogated. By author's opinion, this surplus of safety is one of the reasons for achieving so great success in the service life prolonging of structures, components and joints by the tools of Fracture Mechanics in last few decades..

The same approach to fatigue safety factor determination can be also applied to the known Gerber's parabola, Soderberg's or any other criterion of fatigue failure which takes into account specific material, state of stressing or service conditions.

The formulae derived can be also used for the fatigue assessments of components subjected to variable amplitude or random loadings: it is necessary only to reduce its stress histories to the CA ones. Similarly, in the case of multiaxial loading, it is necessary to reduce the stresses to the equivalent normal one.

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