Finite Deformation: Special Cases

The notes on finite deformation have been divided into three parts:

- General theory (http://imechanica.org/node/538)
- Elasticity of rubber-like materials (http://imechanica.org/node/14146)
- Special cases (http://imechanica.org/node/5065)

The three parts can be read in any order.

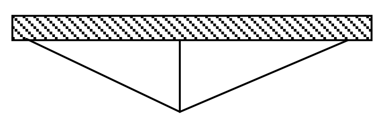
The notes on special cases were initially written for ES 240 Solid Mechanics (http://imechanica.org/node/203).

Go Nonlinear

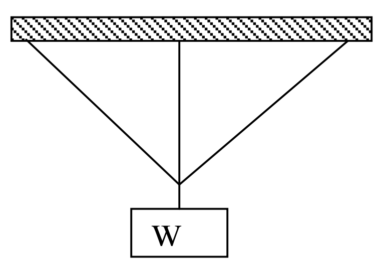
Be wise, linearize. Following this advice of George Carrier, so far in this course we have stayed linear. We have been mostly looking at infinitesimal deformation and Hookean materials. We have mixed the three ingredients: deformation geometry, force balance, and material model. The resulting theory is linear. We have learned fascinating and useful phenomena: stress concentration around a hole, vibration of a beam, refraction of a wave, etc. We have also learned to use commercial finite element code to analyze phenomena with great complexity. We can go on with this linear theory and do a lot more.

We now wish to go nonlinear, hopefully also with wisdom. We will refine the three ingredients by considering finite deformation and non-Hookean materials. We can mix the refined ingredients, or mix a refined ingredient with unrefined ones. For example, as we have already done, a viscoelastic material is non-Hookean, but deformation of such a material can be infinitesimal. We will outline basic ideas of finite deformation, and describe phenomena that show how finite deformation makes difference.

Finite deformation. When a structure deforms, each state of deformation must obey Newton's law. This principle has often been violated in our past work. For example, in analyzing a truss, we have balanced forces in the deformed truss using the shape of the undeformed truss, neglecting the deformation.



reference state



current state

This negligence is often justified on the ground that deformation in most engineering structures is small. You might think that a structure suffering a small strain, say less than 1%, entitles you to neglect the change in shape when you balance forces. A counter example, however, is familiar to you. When a column buckles, the strain in the column is indeed small, but you must enforce equilibrium in the deflected state of the column. The essential point is this:

> *We must enforce Newton's law in every deformed state of a structure, and use this correct principle as a basis to justify any simplification.*

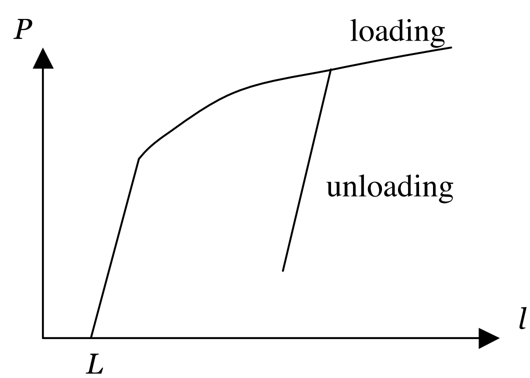
This consideration alone requires us to examine finite deformation, even when the strain is small everywhere in the structure.

Non-Hookean behavior of materials. Hooke's law says that the displacement of a structure is linear in the applied force. This law is an idealization, and contradicts with many daily experiences. Pursuing non-Hookean behavior will take us in many directions. A structure may undergo finite, history-dependent deformation in response to diverse stimuli. Here are some examples.

Nonlinear elasticity. When a force causes a material to deform by a large amount, the displacement may be a nonlinear function of the force. We will talk more about nonlinear elasticity in this set of notes.

Viscoelasticity. We have already looked at time-dependent material behavior, such as viscoelasticity.

Plasticity. After elastic deformation, upon unloading, a metal recovers its shape. After plastic deformation, upon unloading, the metal does not fully recover its shape. Say we apply an axial force to a metal bar, and measure its length. The experimental record of the force-length relation is linear for elastic deformation, and is nonlinear for plastic deformation. During unloading, the metal bar deforms elastically. After plastic loading and elastic unloading, the force-length relation is not one-to-one, but is **history-dependent**. The theory of plasticity will be taught in a separate course.



Deformation in response to diverse stimuli. A material may deform in response to stimuli other than a mechanical force. For example, we have already analyzed deformation caused by change in temperature. Indeed, a material may deform in response to electric field, moisture, light, ionic concentration, etc. We will pick up this fascinating subject in a subsequent course, ES 241, Advanced Elasticity (http://imechanica.org/node/725).

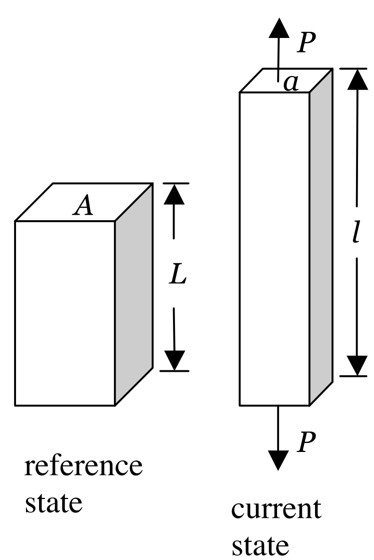
Stress and Strain

Why do we need stress and strain? We proceed with our subject incrementally, beginning with the simplest structure: a bar. When the bar is not subject to any force, the cross-sectional area is A and the length is L. We will call this state the reference state. The bar is then subject to an axial force P, and deforms to a new state, cross-sectional area a and length l. We will call this state the current state. The experimentalist records, among other things, the force as a function of the length.

The force-length curve gives us some idea of the behavior of the material. However, we wish to use the curve to predict the behavior of another structure made of the same material. To do so, we need to construct a material model that is independent of the shape and size of the bar. As a step forward, we divide the elongation by the length of the bar, and divide the force by the cross-sectional area of the bar. This practice is sensible provided the deformation of the bar is homogenous, and the resulting stress-strain relation is independent of the size of the bar.

This practice is economic. In combination with other ingredients, the stress-strain relation can predict the behavior of any structure made of the same material, even when the structure is of a shape other than a bar, and when deformation is inhomogeneous. We can divide the body into many small volumes, and assume that each small volume undergoes homogeneous deformation.

In the following paragraphs, we will first define strain and stress, and describe the stress-strain relations for particular materials. We then analyze several phenomena.



Strain. Define the **engineering strain** by the elongation of the bar in the current state divided by the length of the bar in the reference state:

$$e = \frac{\text{elongation}}{\text{length in the reference state}} = \frac{l - L}{L}.$$

Another type of strain is defined as follows. Deform the material from a current length l by a small amount to $l + \delta l$. Define the increment in the strain, $\delta\varepsilon$, as the increment in the length of the bar divided by the current length of the bar, namely,

$$\delta\varepsilon = \frac{\text{increment in length}}{\text{length in the current state}} = \frac{\delta l}{l}.$$

This equation defines the increment of **natural strain**. Integrating from L to l, we obtain that

$$\varepsilon = \log\left(\frac{l}{L}\right).$$

Yet one more type of strain, the **Lagrange strain**, is defined as

$$\eta = \frac{1}{2}\left[\left(\frac{l}{L}\right)^2 - 1\right].$$

This definition is hard to motivate in one dimension. But if you take the view that any one-to-one function of l/L is a definition of strain, then no motivation is needed.

Indeed, even the ratio l/L itself has a name: the **stretch**. Define the stretch as the length of the bar in the current state divided by the length of the bar in the reference state:

$$\lambda = \frac{\text{length in current state}}{\text{length in reference state}} = \frac{l}{L}.$$

There seems to be no lack of ingenuity to invent yet another definition of strain. All these definitions contain the same information: the ratio l/L. For example, every one of the definitions above is a function of the stretch:

$$e = \lambda - 1,\ \varepsilon = \log\lambda,\ \eta = \frac{\lambda^2 - 1}{2}.$$

Because they are all one-to-one functions, any one definition can be taken to be "basic" and then used to express all the other definitions. For example, we can express all definitions in terms of the engineering strain:

$$\lambda = e + 1,\ \varepsilon = \log(1 + e),\ \eta = \frac{1}{2}e^2 + e$$

When deformation is small, namely, $e << 1$, the three definitions of strain are approximately equal, $\varepsilon \approx \eta \approx e$.

When we call e the engineering strain, we do not mean that e is quick-and-dirty, or unscientific, or unnatural. We just wish to name the quantity $(l - L)/L$. Later on, we will describe motivations for some of the definitions, but these motivations are just elaborate ways to express preferences of individual people. The motivations, however elaborate, should not obscure a simple fact:

you can use any one-to-one function of l / L to define strain.

Stress. When dealing with finite deformation, we must be specific about the area used in defining the stress. Define the **nominal stress**, *s*, as the force applied to the bar in the current state divided by the cross-sectional area of the bar in the reference state:

$$s = \frac{\text{force in the current state}}{\text{area in the reference state}} = \frac{P}{A}.$$

The nominal stress is also known as the engineering stress, or the first Piola-Kirchhoff stress.

Define the **true stress**, σ, as the force in the current state divided by the area in the current state, namely,

$$\sigma = \frac{\text{force in the current state}}{\text{area in the current state}} = \frac{P}{a}.$$

The true stress is also known as the Cauchy stress

Once again, you should not be misled by the names, or influenced by the prejudice of your teachers. The true stress is no truer than the nominal stress. The engineering stress is no less scientific. They are just different definitions of stress, and we need to have different names for them.

Work. When the bar elongates from length l to length $l + \delta l$, the force P does work $P\delta l$. Recall one pair of definitions of stress and strain:

$$P = sA, \quad l = \lambda L.$$

Consequently the work done by the force is

$$P\delta l = ALs\delta\lambda.$$

Since AL is the volume of the bar in the reference state, we note that

$$s\delta\lambda = \frac{\text{increment of work in the current state}}{\text{volume in the reference state}}.$$

We say that the nominal stress and the stretch are *work-conjugate*. Also note that $\delta e = \delta\lambda$, so that the nominal stress is also work-conjugate to the engineering strain.

Recall another pair of definitions of stress and strain:

$$P = \sigma a, \quad \delta l = l\delta\varepsilon.$$

The work done by the force is

$$P\delta l = al\sigma\delta\varepsilon.$$

Since al is the current volume of the bar, we note that

$$\sigma\delta\varepsilon = \frac{\text{increment of work in the current state}}{\text{volume in the current state}}.$$

That is, the true stress is work-conjugate to the natural strain.

Given any definition of strain, we can define its work-conjugate stress. For example, consider the Lagrange strain, η. Subject to an increment in the strain, $\delta\eta$, the force acting on the element does the work. Denote

$$S\delta\eta = \frac{\text{increment of work in the curent state}}{\text{volume in the reference state}}.$$

This expression gives a new definition of stress, S. This new definition does not have a "simpler" interpretation than its status as the work conjugate to the Lagrange strain.

If we are liberal about the definition of strain, without being obsessive about "motivating" each definition, we may as well take a liberal view to call the work conjugate of each definition of strain a definition of stress, and name our definition after a mechanician who can no longer protest.

We can easily invent and name definitions, but all the above definitions have already had names:

- σ: the true stress or the Cauchy stress.
- s: the nominal stress or the first Piola-Kirchhoff stress.
- S: the second Piola-Kirchhoff stress.

Recall the relations among the definitions of strain:

$$e = \lambda - 1, \ \varepsilon = \log \lambda, \ \eta = \frac{\lambda^2 - 1}{2}$$

We obtain the relations among their increments:

$$\delta e = \delta\lambda, \ \delta\varepsilon = \frac{\delta\lambda}{\lambda}, \ \delta\eta = \lambda\delta\lambda .$$

Consequently, the three definitions of stress are related as

$$\sigma a = sA, \quad s = \lambda S .$$

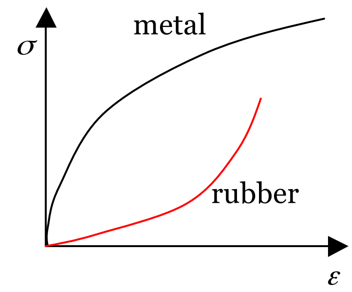
Are these alternative definitions of strain and stress necessary? I have my opinion, and you may have yours. Whatever your opinions are, some of your agony in studying the subject may be alleviated by knowing that textbooks of nonlinear continuum mechanics are full of alternatives at every turn. These alternatives hide behind sophisticated symbols and seductive motivations. You will just have to look beyond them, and focus on matters of substance.

Material models. Using a bar of a given material, our experiment records a force-length curve. We then convert the curve to a stress-strain relation by using some definitions of stress and strain. Any definitions serve the purpose. All that matters is that we tell other people which definitions we have used.

For a **metal** undergoing large, plastic deformation, the stress-strain curve (without unloading) is often fit to a power law in terms of the true stress and the natural strain:

$$\sigma = K\varepsilon^N,$$

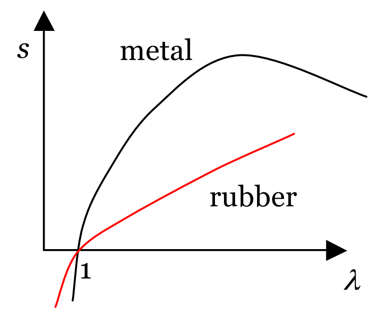
where K and N are parameters to fit experimental data. Some representative values: N = 0.15-0.25 for aluminum, N = 0.3-0.35 for copper, N = 0.45-0.55 for stainless steel. K has the dimension of stress; it represents the true stress at strain $\varepsilon = 1$. Representative values for K are 100 MPa – 1GPa.



For a **rubber**, the stress-strain data may be fit to a relation known as the neo-Hookean model:

$$s = \mu\left(\lambda - \lambda^{-2}\right).$$

Representative values for μ are 0.1 MPa – 10 MPa.



When a material undergoes large deformation, volumetric strain is often negligible compared to tensile strain. Consequently, such a material may be taken to be incompressible.

To see the difference in material behavior, we should use the same definition of stress and strain for both materials. Recall that $\sigma = s\lambda$ (for incompressible materials) and $\varepsilon = \log\lambda$. In terms of the nominal stress and the stretch, the two material models are

$$s = K\frac{(\log\lambda)^N}{\lambda}, \quad \text{for metals}$$

$$s = \mu\left(\lambda - \lambda^{-2}\right), \quad \text{for rubbers}$$

In terms of the true stress and the natural strain, the two material models are

$$\sigma = K\varepsilon^N, \quad \text{for metals}$$

$$\sigma = \mu\left[\exp(2\varepsilon) - \exp(-\varepsilon)\right], \quad \text{for rubbers}$$

This change of variables makes evident a key difference: in tension, the stress in rubbers rises more steeply than in metals. We will return to this difference shortly.

Exercise. Use the 3 ingredients outlined about to obtain the force-deflection relation for the truss sketched in the beginning of the notes. Assuming that all three members of the truss are made of rubber bands, and that deformation is large.

A Metal Bar Forms a Neck, But a Rubber Band Doesn't

Let us mix the newly refined three ingredients to analyze a specific phenomenon: necking. Subject to a tensile force, a metal bar first elongates to a homogenous state of strain. At some level of strain, a small part of the bar thins down preferentially, forming a neck. By contrast, a rubber band under tension usually deforms homogeneously, and does not form a neck. We would like to understand these observations.

Three ingredients of continuum mechanics. Here is a summary of the three ingredients:

- Geometry of deformation: $\varepsilon = \log(l/L)$.
- Balance of forces: $P = \sigma a$.
- Material model: For a metal bar under uniaxial tension, the true stress relates to the natural strain as $\sigma = \sigma(\varepsilon)$. When deformation is large, a material typically changes shape much more than changes volume. We adopt a commonly used a simplification that assumes the volume of the bar to be unchanged during deformation, $AL = al$, or $a = A\exp(-\varepsilon)$.

Mixing the three ingredients, we obtain the force as a function of strain:

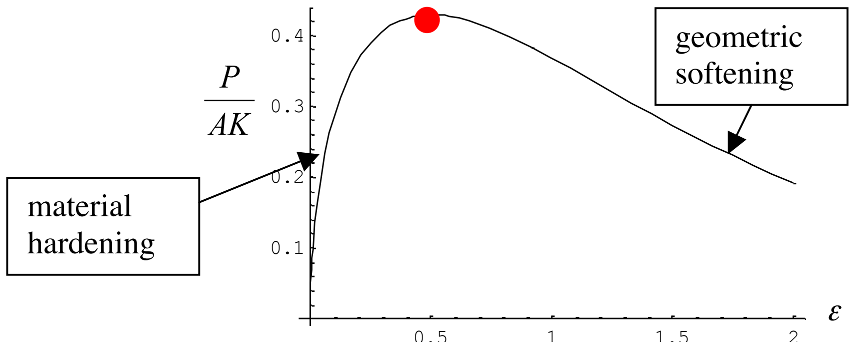
$$P = A\sigma(\varepsilon)\exp(-\varepsilon).$$

This expression relates the applied force P to the strain of the bar once we specify a material model, namely the stress-strain relation $\sigma = \sigma(\varepsilon)$.

Necking in a metal bar. For metals, we use material model $\sigma = K\varepsilon^N$ and obtain that

$$P = AK\varepsilon^N \exp(-\varepsilon).$$

Plot P as a function of ε. In plotting the figure, I've set $N = 0.5$. Observe the two competing factors: material hardening and geometric softening. As the bar elongates, the material hardens, as reflected by the hardening exponent in the stress-strain relation $\sigma = \sigma(\varepsilon)$. At the same time, the elongation reduces the cross-sectional area, an effect known as geometric softening. For small deformation, $P \to 0$ as $\varepsilon \to 0$; material hardening prevails, and the force increases as the bar elongates. For large deformation, so long as the stress-strain relation $\sigma(\varepsilon)$ increases slower than $\exp(\varepsilon)$, $P \to 0$ as $\varepsilon \to \infty$, geometric softening prevails, and the force drops as the bar elongates.



To determine the peak force, note that

$$\frac{dP}{d\varepsilon} = A\exp(-\varepsilon)\left[\frac{d\sigma(\varepsilon)}{d\varepsilon} - \sigma\right].$$

Consequently, the force P peaks when the true stress equals the tangent modulus:

$$\frac{d\sigma(\varepsilon)}{d\varepsilon} = \sigma(\varepsilon).$$

This equation, known as the **Considère condition**, determines the strain at which the force peaks. For the power-law material, $\sigma(\varepsilon) = K\varepsilon^N$, the force peaks at the strain

$$\varepsilon_c = N.$$

When the metal bar is pulled beyond the critical strain, deformation becomes inhomogeneous, with a segment of the bar elongates at a higher strain than the rest of the bar. That is, a neck forms in the bar. After a neck forms, the field of stress in the bar is no longer uniaxial, but is triaxial. Such an inhomogeneous deformation can be analyzed by using the finite element method. For example, the necking process can be studied by using ABAQUS.

- A. Needleman, A numerical study of necking in circular cylindrical bar, Journal of the Mechanics and Physics of Solids, 20, 111 (1972).

A rubber band under tension does not form a neck. For a rubber band, assume the material is Neo-Hookean:

$$\sigma = \mu\left(\lambda^2 - \lambda^{-1}\right) = \mu[\exp(2\varepsilon) - \exp(-\varepsilon)].$$

Thus, at a large tensile strain, the true stress increases exponentially with the natural strain, so that the force $P = A\sigma(\varepsilon)\exp(-\varepsilon)$ always increases with the strain. The rubber band will not form a neck under uniaxial tension.

Exercise. Argue that necking sets in at a stretch where nominal stress is maximum. Thus, in terms of the nominal stress and the stretch, the Considère condition is

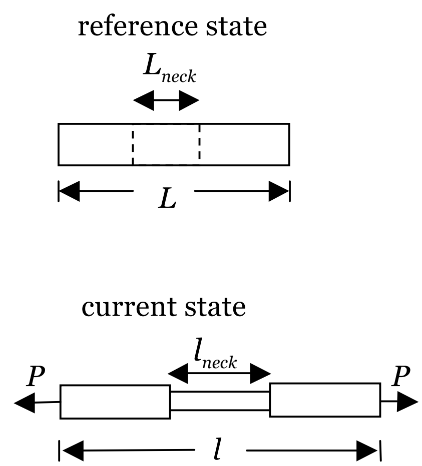
$$\frac{ds(\lambda)}{d\lambda} = 0.$$

Apply this condition to metals and rubbers. This exercise shows that the same physical condition may take different forms when different definitions of stress and strain are used. In this case, the use of nominal stress simplifies the discussion.

Long-neck model. This model assumes the neck to be long compared to the diameter of the bar, so that the deformation in the neck is homogenous, characterized by the stretch λ_{neck}. The deformation in the remainder of the bar is also homogeneous, characterized by another stretch, λ_0. The neck stretches more than the rest of the bar:

$$\lambda_{neck} > \lambda_0.$$

Axial force in the neck is the same as the applied force *P*. Consequently, the nominal stress in the neck equals the nominal stress in the rest of the bar:



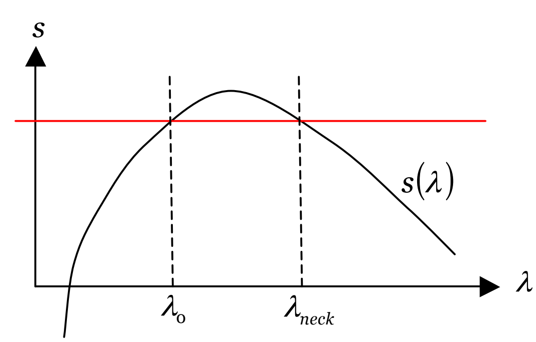
$$s(\lambda_{neck}) = s(\lambda_0).$$

When the stress-stretch curve $s(\lambda)$ is not monotonic, necking is possible.

Let L_{neck} be the length of the neck in the reference state. The length of the bar in the current state is

$$l = L_{neck}\lambda_{neck} + (L - L_{neck})\lambda_0.$$

This idealized model enables us to interpret several experimental observations.



Effect of strain rate on necking. At an elevated temperature, glass flows and can be drawn into a thin fiber without forming necks. The rate-dependent deformation also has significant effect on the stability of metals even at the room temperature. Metals appear to be harder when they are loaded at a higher strain rate. A material model capturing this effect of strain rate is

$$\sigma = K\varepsilon^N \left(\frac{d\varepsilon}{dt} \right)^m ,$$

where m is the strain-rate hardening exponent. Experiments indicate that this strain-rate sensitivity can significantly increase the elongation of a metal before fracture. This observation may be understood as follows. Consider an imperfect bar, with a segment thinner than the rest of the bar. When the bar is pulled, the thinner region has a higher strain rate, and is therefore harder, than the rest of the bar. This effect of strain rate counteracts necking.

The axial force is $P = \sigma a$. Due to the assumption of incompressibility, the cross-sectional area relates to a the strain as $a = AL/l = A\exp(-\varepsilon)$. Thus, the axial force relates to the strain as

$$P = KA\exp(-\varepsilon)\varepsilon^N \left(\frac{d\varepsilon}{dt} \right)^m .$$

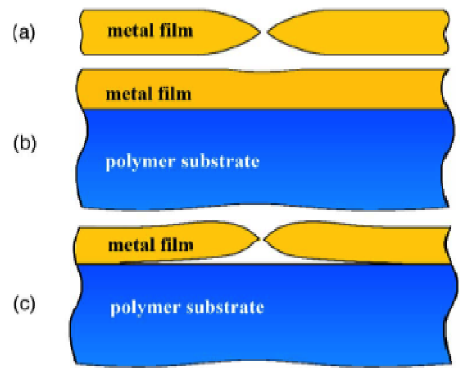
Now apply this expression to the imperfect bar. In the unstressed state, let the cross-sectional area of the thinner segment of the bar be A_{neck}, and let the cross-sectional area of the remaining parts of the bar be A_o. In the current state, when the bar is subject to the axial force P, everywhere in the bar sustains the same axial force. Denote the strain in the thinner segment of the bar by ε_{neck}, and the strain in the rest parts of the bar by ε_o. The equality of the axial force in the bar requires that

$$\left(\frac{A_{neck}}{A_o} \right)^{\frac{1}{m}} \exp\left(-\frac{\varepsilon_{neck}}{m} \right) \varepsilon_{neck}^{N/m} d\varepsilon_{neck} = \exp\left(-\frac{\varepsilon_o}{m} \right) \varepsilon_o^{N/m} d\varepsilon_o .$$

This equation can be used to plot ε_{neck} against ε_o. See descriptions of the results and comparison with experiments in the following paper.

- J.W. Hutchinson and K.W. Neale, Influence of strain-rate sensitivity on necking under uniaxial tension, Acta Metallurgica 25, 839-846 (1977). http://www.seas.harvard.edu/hutchinson/papers/340.pdf

Freestanding thin metal wires and thin metal films fracture at small strains. When a freestanding film of a plastically deformable metal is subjected to a tensile load, the film ruptures by strain localization, by forming a neck within a narrow region, of a wid comparable to the thickness of the film. The strain is large within the neck, but is small elsewhere in the film. Because the f has an extraordinarily large length-thickness ratio, the net elongation of the film upon rupture is small, typically less than a few percent.



Let L_{neck} be the length of the neck in the reference state. The length of the bar in the current state is

$$l = L_{neck}\lambda_{neck} + \left(L - L_{neck}\right)\lambda_0 .$$

The apparent stretch of the bar is

$$\frac{l}{L} = \frac{L_{neck}}{L}\lambda_{neck} + \left(1 - \frac{L_{neck}}{L}\right)\lambda_0 .$$

The neck is highly localized: the length of the neck is typically on the order of the thickness of the film,

$$L_{neck} \approx H .$$

For a thin film, the length of the neck is much shorter than the length of the bar, $L_{neck} / L << 1$. Consequently, the apparent stretch of the bar is $l / L \approx \lambda_0$.

A thin metal film bonded to a polyimide substrate can be stretched beyond 50%. For a metal film well bonded to a polymer substrate, the substrate can delocalize strain, so that the metal film can elongate indefinitely, only limited by the rupture of the polymer substrate. Rupture of a thin metal film by strain localization: (a) Freestanding film: local elongation leads to rupture; (b) Supported film: local elongation suppressed by substrate; (c) Debonding assists in rupture.

- N.S. Lu, X. Wang, Z.G. Suo and J. J. Vlassak, Metal films on polymer substrates stretched beyond 50%. Applied Physics Letters 91, 221909 (2007). http://www.seas.harvard.edu/suo/papers/201.pdf

Inflation of a Balloon

When a spherical balloon is in the unstressed state, the radius of the balloon is R, and the thickness of its membrane is H. When the balloon is subject to a pressure p, the radius of the balloon expands to r, and the thickness of the membrane reduces to h. We would like to relate the radius r and the pressure p.

Kinematics. In the current state, the hoop stretch of the balloon is

$$\lambda = r / R.$$

Force balance. The stress normal to the membrane is between p and zero, and is small compared to the hoop stress σ. Consequently, we assume that the membrane is in a state of equal-biaxial stress. Draw the free-body diagram of a half balloon in the current state, and balance the forces: $2\pi r h \sigma = \pi r^2 p$. Thus

$$p = \frac{2h}{r}\sigma.$$

Because $r >> h$, we confirm that $\sigma >> p$.

Material model. The material is taken to be incompressible, so that $hr^2 = HR^2$, or

$$h = H\lambda^{-2}.$$

Suppose that we have tested the material, and obtained the stress-stretch relation in the state of equal-biaxial stress:

$$\sigma = \mu\lambda^{-4}\left(\lambda^6 - 1\right).$$

Mix the these ingredients together, and we obtain that

$$p = \frac{2H\mu}{R}\lambda^{-7}\left(\lambda^6 - 1\right).$$

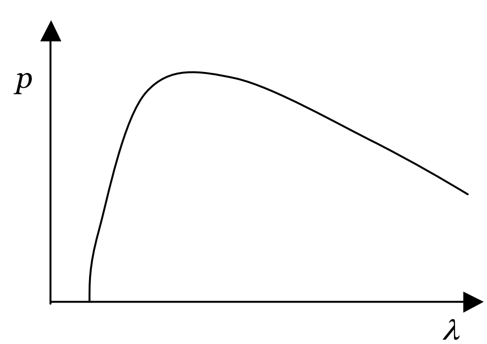
This is the desired relation between the pressure and the stretch. Sketch the function $p(\lambda)$. When the balloon is unstretched, $\lambda = 1$, the pressure is zero, $p(1) = 0$. As the balloon expands, the material stiffens, but the membrane thins. For the neo-Hookean material the above calculation shows that $p(\infty) = 0$. When the stretch is large enough, the geometric thinning prevails over material stiffening, and the pressure reaches a peak. This peak is reached at the critical radius

$$\lambda_c = 7^{1/6} \approx 1.38.$$

The peak pressure is

$$p_c = 1.2\mu H / R.$$

When the pressure approaches this peak value, this model predicts that the balloon will rupture.



The critical radius above seems to be too small compared to our experience with balloons. Perhaps two reasons account for this discrepancy. First, when we blow a balloon, our lungs cannot supply air at a constant pressure. When the balloon expands significantly, the pressure drops. Second, when the stretch is large, the rubber may stiffen significantly, at a rate much higher than that described the neo-Hookean model.

- I. Muller and H. Struchtrup, Inflating a rubber balloon, Mathematics and materials of Solids 7, 569-577 (2002).
- H. Alexander, Tensile instability of initially spherical balloons. Int. J. Engng. Sci 9, 151-160 (1971).
- A. Needleman, Inflation of spherical rubber balloons, International Journal of Solids and Structures 13, 409-421 (1977).
- D.K. Bogen and T.A. McMahon, Do cardiac aneurysms blow out? Biophysics Journal 27, 301-316 (1979).

Exercise. When a spherical balloon is subject to a constant pressure, allow the balloon to oscillate by changing its radius with time. Determine the natural frequency. When does the natural frequency vanish? Link this condition to the condition of instability.

Exercise. A spherical balloon is inflated by a pressure. Calculate the pressure as a function of the radius, and plot your result in a dimensionless form. The material obeys the Gent model. Plot the curve for two cases: $J_{\lim} = 50,100$. Discuss your results. Under what experimental conditions would you expect a snap-through instability?

Cavitation

Experimental setups. Describe the following experiments

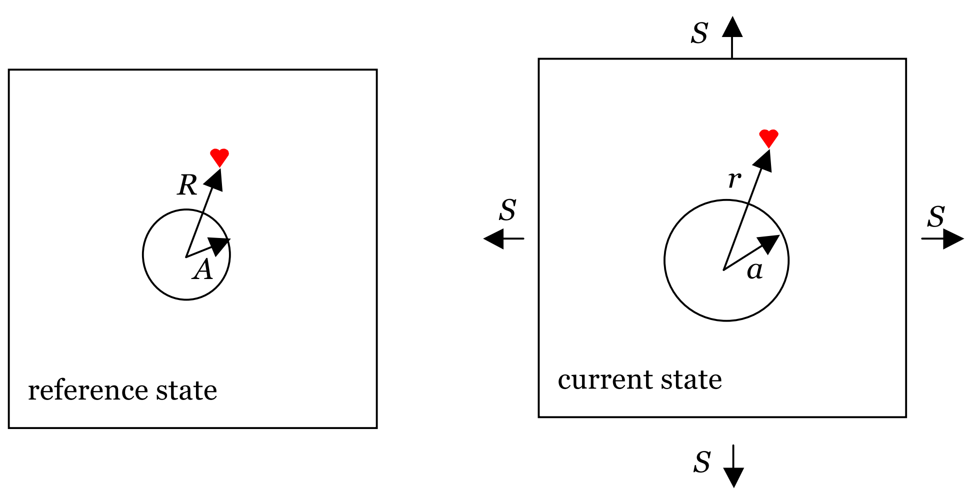
- A.N. Gent and P.B. Lindley. Internal rupture of bonded rubber cylinders in tension. Proc. R. Soc. A 249, 195-205 (1959).
- M.F. Ashby, F.J. Blunt and M. Bannister, Flow characteristics of highly constrained metal wires. Acta Metallurgica 37, 1847-1857 (1989).
- S. Kundu and A.J. Crosby, Cavitation and fracture behavior of polyacrylamide hydrogels, Soft Matter 5, 3963–3968 (2009).

Model. Consider a small spherical cavity in a large block of a material. In the reference state, the material is under no stress, and the radius of the cavity is A. In the current state, the body remote from the cavity is subject to a triaxial tensile stress S, and the radius of the cavity becomes a. For a brittle material, the cavity concentrates stress, so that a crack may emanate from the cavity. In such a case, deformation is small when the stress reaches a critical value, so that we can analyze the problem assuming that the material is linearly elastic and deformation is infinitesimal. Formulated this way, the problem is known as the Lame problem, as we have seen before. The key result of the analysis is that the hoop stress at the surface of cavity is 3/2 times the remote stress.

For a material capable of large deformation, such as a ductile metal or a rubber, however, the cavity may cause another mode of failure. Under the hydrostatic tension remote from the cavity, the cavity may expand indefinitely when the applied stress reaches some finite value, a phenomenon known as cavitation. To study this phenomenon, we need to determine the radius of the cavity as a function of the remote tension. The Lame solution predicts that the radius of cavity increases linearly with the applied stress. We next analyze the phenomenon allowing finite deformation, and show that the Lame solution is wrong. The three ingredients of solid mechanics take the following form.

Geometry of deformation. Consider a material particle in the body. When the body is in the reference state, the material particle is at distance R from the center of the cavity. When the body is in the current state, the material is at distance r from the center of the cavity. The geometry of deformation is fully described by the function

$$r = r(R).$$



Consider a set of material particles. In the reference state, the set of material particles forms a circle of radius R, and the circumference of the circle is $2\pi R$. In the current state, the same set of material particles forms a circle of radius r, and the circumference of the circle is $2\pi r$. Consequently, the hoop stretch is

$$\lambda_\theta = \frac{r(R)}{R}.$$

Next consider two material particles. In the reference state, the two particles lie on the same radius, distance R and $R + dR$ from the center of the cavity. In the current state, the two particles still lie on the same radius, but are at distance $r(R)$ and $r(R + dR)$ from the center of the cavity. Consequently, the radial stretch is

$$\lambda_r = \frac{r(R + dR) - r(R)}{dR} = \frac{dr(R)}{dR}.$$

Balance of Forces. In the current state, the state of stress in the body remote from the cavity is equal-triaxial, but the state of stress in the body near the cavity is equal-biaxial. That is, the state of stress in the body is inhomogeneous. A material particle at location r is subject to a state of triaxial stress:

$$\sigma_r(r),\ \sigma_\theta(r) = \sigma_\varphi(r).$$

We balance force for a material particle in the current state, so that

$$\frac{d\sigma_r(r)}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0.$$

Material model. We assume that the material is incompressible. This assumption markedly simplifies the problem. In the reference state, the volume of the shell between radii A and R is $(4\pi/3)(R^3 - A^3)$. In the current state, the shell deforms and the radii become a and r, and the volume of the shell is $(4\pi/3)(r^3 - a^3)$. Because the material is assumed to be incompressible, the volume of the shell in the reference state equals that in the current state:

$$r^3 - a^3 = R^3 - A^3.$$

Consequently, once we determine the radius of the cavity in the current state, a, we will know the entire field of deformation $r(R)$. The assumption of incompressibility reduces the deformation of the body to a single degree of freedom: the radius of the cavity a.

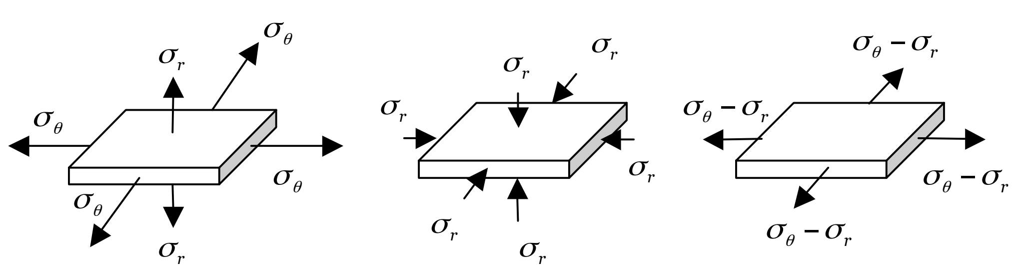
Suppose that we have tested the material under the equal-biaxial state of stress. When a sample of the material is stretched by equal-biaxial state of stress, $(\sigma, \sigma, 0)$, the material deforms into a state of stretch $(\lambda, \lambda, \lambda^{-2})$. Because the material is taken to be incompressible, the state of stretch under equal-biaxial stress is characterized by a single parameter, λ. Let us say that the experimental measurement gives us a stress-stretch curve in the state of equal-biaxial stress, written as

$$\sigma = g(\lambda).$$

A material particle is under a triaxial state of stress: one radial component and two hoop components, $(\sigma_r, \sigma_\theta, \sigma_\theta)$. The material particle is also in a triaxial state of stretch, $(\lambda_r, \lambda_\theta, \lambda_\theta)$. Because the material is taken to be incompressible, the radial stretch relates to the hoop stretch as $\lambda_r = \lambda_\theta^{-2}$. Also because the material is taken to be incompressible, superposing a hydrostatic stress on the material particle will not change the state of deformation of the particle. For example, we can superimpose $(-\sigma_r, -\sigma_r, -\sigma_r)$ on the material particle, so that the stress state of the particle becomes $(0, \sigma_\theta - \sigma_r, \sigma_\theta - \sigma_r)$. This is an equal-biaxial state of stress. Thus,

$$\sigma_\theta - \sigma_r = g(\lambda_\theta).$$

Because of the spherical symmetry of this phenomenon, we will only need the stress-strain curve determined in the equal-biaxial state of stress to analyze the process of cavitation.



triaxial stresses Hydrostatic stress biaxial stresses

Mix the three ingredients. Integrating the force-balancing equation $d\sigma_r / dr = 2(\sigma_\theta - \sigma_r)/r$, we obtain that

$$S = 2\int_a^\infty \frac{g(\lambda_\theta)dr}{r}.$$

We have used the boundary conditions: $\sigma_r(a) = 0$ and $\sigma_r(\infty) = S$.

Recall that $\lambda_\theta = r/R$, and that the elastomer is taken to be incompressible, $r^3 - a^3 = R^3 - A^3$. Consequently, r is a function of λ_θ, namely,

$$r = \left(a^3 - A^3\right)^{1/3} \lambda_\theta \left(\lambda_\theta^3 - 1\right)^{-1/3}.$$

In the integral above, we change the variable from r to λ_θ, and we obtain that

$$S = 2\int_{a/A}^{1} \frac{g(\lambda_\theta)d\lambda_\theta}{\lambda_\theta\left(\lambda_\theta^3 - 1\right)}.$$

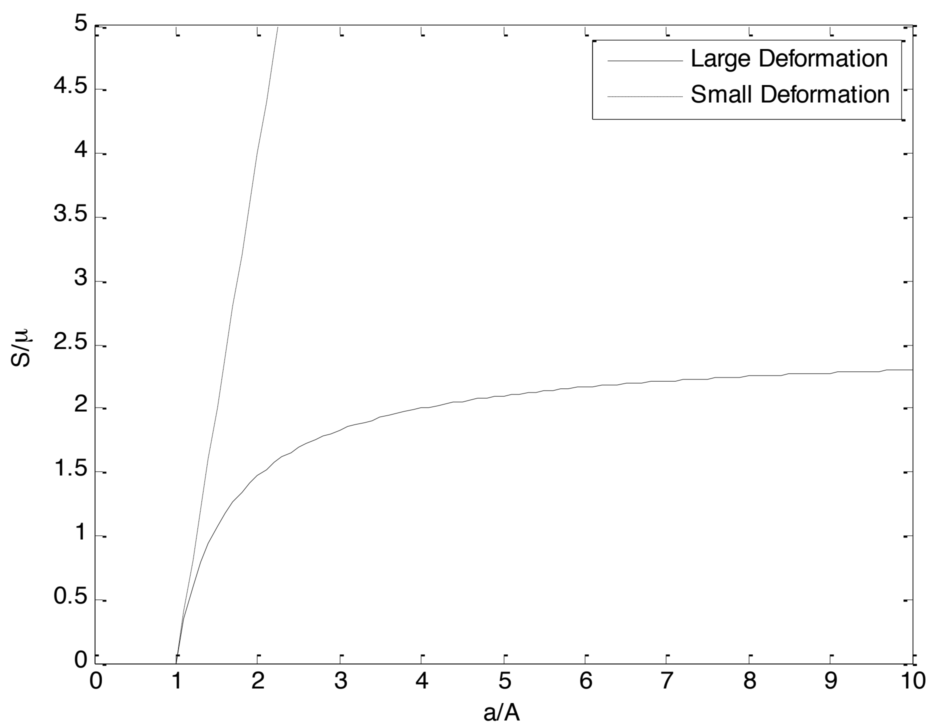
This equation relates the remotely applied stress, S, to the stretch of the cavity, a/A, once the stress-stretch curve $g(\lambda)$ is prescribed.

For the Neo-Hookean material, the stress-stretch relation in the state of equal-biaxial stress is

$$g(\lambda) = \mu\lambda^{-4}\left(\lambda^6 - 1\right).$$

The integral can be evaluated analytically, giving

$$\frac{S}{\mu} = \frac{5}{2} - 2\left(\frac{a}{A}\right)^{-1} - \frac{1}{2}\left(\frac{a}{A}\right)^{-4}.$$



Sketch this result and compare it with the Lame solution, which assumes Hookean material and infinitesimal deformation. The cavity can expand indefinitely when the remote stress is still finite. The remote stress needed to cause the cavity to expand indefinitely ($a/A \to \infty$) is called the **cavitation limit**, and is

$$S_c = \frac{5}{2}\mu$$

for a Neo-Hookean material.

- For a review of cavitation, see the iMechanica journal club: Cavitation in Soft Solids, Oscar Lopez-Pamies, May 2010 (http://imechanica.org/node/8131).
- A.N. Gent. Cavitation in rubber: A cautionary tale. Rubber Chemistry and Technology 63, G49-G53 (1991).
- Jian Zhu, Tiefeng Li, Shengqiang Cai, and Zhigang Suo. Snap-through expansion of a gas bubble in an elastomer (http://www.seas.harvard.edu/suo/papers/239.pdf). The Journal of Adhesion 87, 466-481 (2011).

Exercise. Plot the hoop stress at the surface of the cavity as function of a/A. Compare the result with the Lame solution.

Exercise. Plot S as a function of a/A for a power-law material. Numerical integration might be needed. Assume that values of the exponent. This problem was studied in Huang, Y., Hutchinson, J.W., Tvergaard, V., " Cavitation Instabilities in Elastic-Plastic Solids." J. Mech. Phys. Solids, 39, 223-241 (1991). http://www.seas.harvard.edu/hutchinson/papers/411.pdf

Exercise. Study the problem for a cavity in a sphere of material of finite radius.

Exercise. Study the phenomenon of cavitation under the plane strain conditions.

Exercise. In the above formulation, we have used the equilibrium equation formulated in terms of Cauchy stresses as functions of r. Derive the equilibrium equation in terms of the nominal stresses as functions of R. Solve the problem using this alternative formulation.

Exercise. Read the paper S. Kundu and A.J. Crosby, Cavitation and fracture behavior of polyacrylamide hydrogels, Soft Matter 5, 3963–3968 (2009). Model a bubble as a sphere, and include the effect of the surface energy of the bubble by using the Laplace pressure. Compare your result with Equation (5) in the paper. Discuss Figs. 2(b) and 2(c) in the context of the experiment described in the paper.

A Cube under Equal-triaxial Forces

This example was taken from R.S. Rivlin, Large elastic deformation of isotropic materials. II. Some uniqueness theorems for pure homogeneous deformation. Philos. Trans. Roy. Soc. Lond. A 240, 491-508, 1948. Consider a cube of an incompressible material subject to a hydrostatic stress. An obvious state of equilibrium is that the cube remains undeformed. Can the cube deform into a rectangular block?

Hydrostatic true stresses. Let the undeformed block be a cube of side *L*. Subject to the force, the cube deforms a rectangular shapes of sides $\lambda_1 L$, $\lambda_2 L$, and $\lambda_3 L$. The material is taken to be incompressible,

$$\lambda_1 \lambda_2 \lambda_3 = 1,$$

and neo-Hookean

$$\sigma_1 - \sigma_3 = \mu\left(\lambda_1^2 - \lambda_3^2\right),$$

$$\sigma_2 - \sigma_3 = \mu\left(\lambda_2^2 - \lambda_3^2\right).$$

The three equations determine the three stresses once the applied stresses $\sigma_1, \sigma_2, \sigma_3$ are known. The neo-Hookean model is developed in notes on rubber elasticity (http://imechanica.org/node/14146).

If the applied hydrostatic stresses are given as true stresses, $\sigma_1 = \sigma_2 = \sigma_3$, the above equations give a unique solution:

$$\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Equal-triaxial forces. The situation is more interesting if, instead of fixed true stresses, fixed forces of equal magnitude are applied to the cube in the three directions. The undeformed state is still a state of equilibrium, but another solution becomes possible. Let P be the dead load on the three faces of the block. In the current state, the true stresses in the three directions become

$$\sigma_1 = \frac{P}{L^2 \lambda_2 \lambda_3} = \lambda_1 \frac{P}{L^2},$$

$$\sigma_2 = \frac{P}{L^2 \lambda_3 \lambda_1} = \lambda_2 \frac{P}{L^2},$$

$$\sigma_3 = \frac{P}{L^2\lambda_1\lambda_2} = \lambda_3\frac{P}{L^2}.$$

The stress-stretch relations take the form

$$\frac{P}{L^2}\left(\lambda_1 - \lambda_3\right) = \mu\left(\lambda_1^2 - \lambda_3^2\right),$$

$$\frac{P}{L^2}\left(\lambda_2 - \lambda_3\right) = \mu\left(\lambda_2^2 - \lambda_3^2\right).$$

These two equations, together with the condition for incompressibility $\lambda_1\lambda_2\lambda_3 = 1$, determine the three stresses. A trivial solution is $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

We look for other solutions. If λ_3 is distinct from λ_1 and λ_2, we can eliminate similar factors from the above equations and find that

$$\frac{P}{L^2\mu} = \lambda_1 + \lambda_3,$$

$$\frac{P}{L^2\mu} = \lambda_2 + \lambda_3.$$

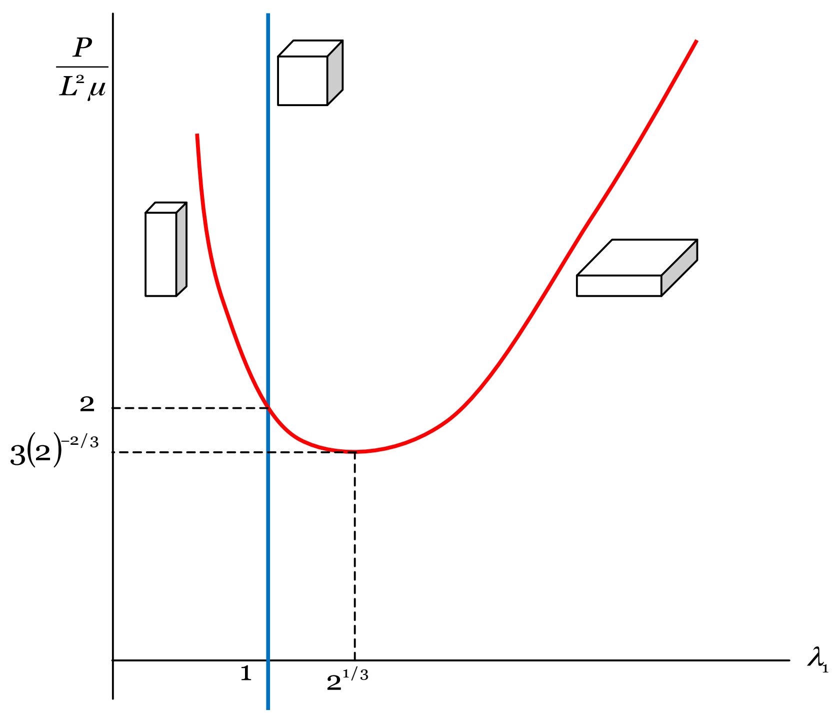
These equations imply that $\lambda_1 = \lambda_2$: the block is in a state of equal-biaxial stretch. Consequently, the solution takes the form

$$\frac{P}{L^2\mu} = \lambda_1 + \frac{1}{\lambda_1^2}.$$

This solution, along with the trivial solution, is plotted in the plane spanned by λ_1 and $P/\left(L^2\mu\right)$. The stretch λ_1 describes the state of the system, and the dimensionless load $P/\left(L^2\mu\right)$ represents a variable parameter. The states of equilibrium change with the variable parameter. Such a diagram is known as the *bifurcation diagram*.

Depending on value of the parameter $P/\left(L^2\mu\right)$, the system may have different numbers of states of equilibrium. When $-\infty < P/\left(L^2\mu\right) < 3\left(2\right)^{-2/3}$, the system has one state of equilibrium, the trivial state. When $P/\left(L^2\mu\right) = 3\left(2\right)^{-2/3}$, the system has two states of equilibrium. When $3\left(2\right)^{-2/3} < P/\left(L^2\mu\right) < 2$, the system has three states of equilibrium. When $P/\left(L^2\mu\right) = 2$, the system has two states of equilibrium. When $P/\left(L^2\mu\right) > 2$, the system has three states of equilibrium. The two points in the bifurcation diagram, $\left(2^{1/2}, 3(2)^{-2/3}\right)$ and $(1, 2)$, are known as *points of bifurcation*, where states of equilibrium change qualitatively.

Incidentally, the values of bifurcation, $P/\left(L^2\mu\right) = 3(2)^{-2/3}$ and $P/\left(L^2\mu\right) = 2$, are both smaller than the cavitation limit 2.5. Thus, the theory predicts that the Rivlin instability precedes the cavitation limit. This prediction, however, has not been examined experimentally. In the experiment of Gent and Lindley (Proc. R. Soc. A 249, 195-205, 1959), a layer of rubber is bonded between two metallic plates. When the plates are pulled in tension, the rubber is in a state of triaxial tension. When the tensile stress exceeds a critical value, cavities in the rubber burst. In this experimental setup, the metallic plates prevent the rubber from deforming significantly. This constraint perhaps suppresses the Rivlin instability.



Exercise. No cube is perfect in reality. To illustrate how imperfection affects the bifurcation diagram, we model an imperfect cube by a rectangular block of three sides, L, L and $\lambda_1\lambda_2\lambda_3 = 1$, where ξ is a dimensionless number of a small magnitude. The rectangular block is made of a neo-Hookean material of shear modulus μ, and is subject to equal-triaxial forces of magnitude P. Plot all states of equilibrium on the plane of axes of $P/\left(L^2\mu\right)$ and λ_1. Assume two values of imperfection: $\xi = +0.01, -0.01$. Discuss your results.

Bifurcation Diagram

Equations of state. Consider a set of nonlinear algebraic equations:

$$\mathbf{f}\left(\mathbf{u},\alpha\right) = \mathbf{0}.$$

Here $\mathbf{u}$ and $\mathbf{f}$ have n components, and α is a single parameter. Once the value of α is prescribed, $\mathbf{f}\left(\mathbf{u},\alpha\right) = \mathbf{0}$ consists of n equations for n unknown components of $\mathbf{u}$. The pair $\left(\mathbf{u},\alpha\right)$ that satisfies $\mathbf{f}\left(\mathbf{u},\alpha\right) = \mathbf{0}$ is called a solution $\mathbf{u}$ associated with a value of α. For a given value of α, the equations $\mathbf{f}\left(\mathbf{u},\alpha\right) = \mathbf{0}$ may have no solution, or one solution, or multiple solutions.

Such equations often arise in studying a system in equilibrium under a load. The set of n components of $\mathbf{u}$ represents the state of the system, α represents the load applied to the system, and the set of n equations $\mathbf{f}\left(\mathbf{u},\alpha\right) = \mathbf{0}$ represents the condition of equilibrium or equations of state. The pair $\left(\mathbf{u},\alpha\right)$ satisfying the equations $\mathbf{f}\left(\mathbf{u},\alpha\right) = \mathbf{0}$ represents a state of equilibrium $\mathbf{u}$ in response to a value of load α. For a given value of the load, the system may have no state of equilibrium, or one state of equilibrium, or multiple states of equilibrium.

All examples given above take this form. In the case of the cube, the three stretches describe the state of cube (the system), the applied force is the loading parameter, and the material model gives the equations of state. In the case of cavitation, a differential equation is obtained. But upon integration, we obtain an algebraic equation between the applied stress (the load) and the radius of the cavity (the state of the system).

More generally, a boundary-value problem in elasticity can be solved by the finite element method. A body is discretized into elements, a set of nodal displacements describes the state of the body, and the balance of forces of all the nodes leads to a set of algebraic equations. If the body is subject to multiple loads, we can vary one load, while fix all the other loads.

A branch of solutions. Let $\mathbf{u}$ be a particular solution for a given value of α. The pair $(\mathbf{u},\alpha)$ satisfies the equations of state:

$$\mathbf{f}(\mathbf{u},\alpha)=\mathbf{0}.$$

When the value of α changes slightly to $\alpha+d\alpha$, we may find another solution $\mathbf{u}+d\mathbf{u}$. The pair $(\mathbf{u}+d\mathbf{u},\alpha+d\alpha)$ also satisfies the equations of state:

$$\mathbf{f}(\mathbf{u}+d\mathbf{u},\alpha+d\alpha)=\mathbf{0}.$$

A comparison of the above two equations, with the aid of the Taylor expansion, gives that

$$\sum_{j=1}^{n}\frac{\partial f_i(\mathbf{u},\alpha)}{\partial u_j}du_j+\frac{\partial f_i(\mathbf{u},\alpha)}{\partial \alpha}d\alpha=0.$$

For a given state of equilibrium $\mathbf{u}$ in response to a given value of α, the partial derivatives $\partial f_i(\mathbf{u},\alpha)/\partial u_j$ form a constant matrix, and the partial derivatives $\partial f_i(\mathbf{u},\alpha)/\partial \alpha$ form a constant column. Consequently, the above expression is a set of linear algebraic equations for $d\mathbf{u}$, for a given value of $d\alpha$. So long as the matrix $\partial f_i(\mathbf{u},\alpha)/\partial u_j$ is not singular, the above linear algebraic equation gives a unique solution of $d\mathbf{u}$ for a given value of $d\alpha$.

Thus, staring with one solution $\mathbf{u}$ for a given value of α, we may obtain a sequence of solutions as α changes. The sequence is known as a branch of solutions. For a system subject to a load, a branch of solutions represents a sequence of states of equilibrium as the load changes.

Bifurcation diagram. The n components of $\mathbf{u}$ and the parameter α together can be used as the axes of a $(n+1)$-dimensional space. A state of equilibrium in response to a given value of the load is noted as a pair $(\mathbf{u},\alpha)$, represented by a point in the $(n+1)$-dimensional space. When the load changes, the system goes through a sequence of states of equilibrium, represented by a curve in the $(n+1)$-dimensional space. The nonlinear equations $\mathbf{f}(\mathbf{u},\alpha)=\mathbf{0}$ may have multiple branches of solutions, each branch being a distinct curve in the $(n+1)$-dimensional space.

Whenever a system requires more than one number to describe its state, drawing the $(n+1)$-dimensional space on a piece of paper becomes impractical.

To illustrate branches of solutions on a plane, we represent the state of the system, **u**, by a single number. The choice of this number is arbitrary. For example, we can choose one component of **u**, say u_1. With this choice, we project the curves in the $(n+1)$-dimensional space to the plane of axes u_1 and α. Such a two dimensional diagram is known as the bifurcation diagram.

We have seen a large number of bifurcation diagrams: the force-displacement curve for a tensile bar, the pressure-volume curve for an inflating balloon, the stress-radius curve for a cavity, and stress-stretch curve for a block.

An active field of research. All these ideas are ancient. What is new is to teach these ideas to the computer, and teach qualitative consequences of these ideas to ourselves. The development in recent decades has resulted in a bag of tools in mathematics, known as bifurcation theory. Active research is ongoing to integrate bifurcation theory with software, to discover new phenomena of bifurcation, and to find new applications.

To learn mathematics of bifurcation, here is an excellent book: R. Seydel, Practical Bifurcation and Stability Analysis, 3rd edition, Springer 2010.

Free Energy and Stability

When a set of nonlinear algebraic equations have multiple solutions, the algebraic equations by themselves cannot tell us which solutions are stable. One way to establish stability is through thermodynamics.

Helmholtz free energy. A bar, length L and cross-sectional area A in the unstressed state, is stretched to the current state by force P to length l and cross-sectional area a. In the current state, when the bar extends from length l to length $l+\delta l$, the force P does work

$$P\delta l .$$

The bar is in thermal equilibrium with a heat reservoir of a constant temperature. Let F be the Helmholtz free energy of the bar. Recall the definition of the Helmholtz free energy (http://imechanica.org/node/4878). When the force P equilibrates with the bar, the work done by the force equals the change in the Helmholtz free energy F of the bar, namely,

$$\delta F = P\delta l .$$

Recall the definitions of the nominal stress s and the stretch λ:

$$P = sA, \quad l = \lambda L .$$

Define the free-energy density by

$$W = \frac{\text{energy in the current state}}{\text{volume of the reference state}}.$$

Divide the equation $\delta F = P\delta l$ by the volume of bar in the reference state, AL, and we obtain that

$$\delta W = s\delta\lambda .$$

We can measure experimentally the nominal stress as a function of stretch, $s(\lambda)$, and then integrate the curve $s(\lambda)$ to obtain $W(\lambda)$. Alternatively, we can obtain an expression of $W(\lambda)$ from some theoretical considerations, and then obtain $s(\lambda)$ by

$$s(\lambda) = \frac{dW(\lambda)}{d\lambda}.$$

For the neo-Hookean model, $s(\lambda) = \mu(\lambda - \lambda^{-2})$. Integrating $s(\lambda)$, we obtain that

$$W(\lambda) = \frac{1}{2}\mu(\lambda^2 + 2\lambda^{-1} - 3).$$

The constant of integration is fixed by following the convention that the free energy is set to be zero when the material is unstressed, $W(1) = 0$.

For a nonlinearly elastic material obeying the power law, $\sigma = K\varepsilon^N$, we have $s = K(\log\lambda)^N \lambda^{-1}$, so that

$$W(\lambda) = \frac{K}{N+1}(\log\lambda)^{N+1}.$$

Helmholtz free energy. A bar, of length L and cross-sectional area A in the reference state, is pulled in the current state by a force P. When the force is fixed, the bar may reach a state of equilibrium, of length l. The stretch is defined by $\lambda = l/L$. The material is taken to be elastic, characterized by the free-energy function $W(\lambda)$. What is the value of the stretch in equilibrium? Is the state of equilibrium stable?

The Helmholtz free energy of the bar is $ALW(\lambda)$. Think of the fixed force P as a weight hanging to the bar. The Helmholtz free energy of the weight is the same as the potential energy of the weight, $-PL(\lambda - 1)$. The bar and the weight together form a composite thermodynamic system. The Helmholtz free energy of the composite is the sum of the Helmholtz free energy of the bar and that of the weight:

$$\Pi = ALW(\lambda) - PL(\lambda - 1).$$

This free energy is a function of the stretch, $\Pi(\lambda)$.

Incidentally, because the independent variable of the function $\Pi(\lambda)$ is not the applied force, we should *not* call Π the Gibbs free energy. In mechanics, the quantity is sometimes called the potential energy of the system, with the meaning of system left unspecified. It is perhaps better to call what it is: the Helmholtz free energy of the composite of the bar and the weight.

The composite—the bar and the weight—interacts with the rest of the world in one way: exchanging energy by heat. The temperature of the surroundings is held fixed. According to thermodynamics, of all values of the stretch, the equilibrium value minimizes the Helmholtz free energy of the composite, $\Pi(\lambda)$. Let $\delta\lambda$ be a small variation. Note the Taylor series:

$$\Pi(\lambda + \delta\lambda) - \Pi(\lambda) = AL\left[\frac{dW(\lambda)}{d\lambda} - \frac{P}{A}\right]\delta\lambda + \frac{1}{2}AL\frac{d^2W(\lambda)}{d\lambda^2}(\delta\lambda)^2.$$

We have retained up to terms quadratic in $\delta\lambda$.

Condition of equilibrium. A state of equilibrium is determined by requiring that

$$\frac{d\Pi(\lambda)}{d\lambda} = 0.$$

This condition of equilibrium is equivalent to

$$\frac{dW(\lambda)}{d\lambda} = \frac{P}{A}.$$

This equation recovers that

$$\frac{dW(\lambda)}{d\lambda} = s.$$

When the force P is prescribed, the nominal stress s is known. When the force equilibrates with the bar, the stretch of the bar, λ, is determined by the above equation, usually a nonlinear algebraic equation. Denote the stretch of a state of equilibrium by λ_{eq}.

Stability of a state of equilibrium. The state of equilibrium is stable against arbitrary small perturbation in stretch if the function $\Pi(\lambda)$ is a minimum at λ_{eq}. This condition for stability is equivalent to

$$\frac{d^2\Pi(\lambda)}{d\lambda^2} > 0,$$

where the stretch is evaluated at the state of equilibrium, λ_{eq}.

The condition for stability is equivalent to

$$\frac{d^2W(\lambda)}{d\lambda^2} > 0,$$

where the stretch is evaluated at the state of equilibrium, λ_{eq}.

The critical condition is

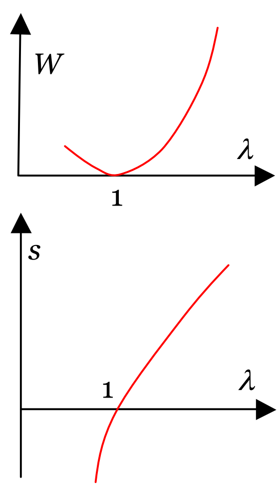
$$\frac{d^2W(\lambda)}{d\lambda^2} = 0,$$

or

$$\frac{ds(\lambda)}{d\lambda} = 0.$$

This critical condition is reached at the peak of the curve $s(\lambda)$.

Considère condition reconsidered. For a neo-Hookean material, the free-energy function is



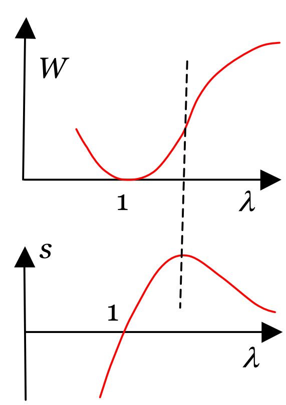
$$W(\lambda) = \frac{1}{2}\mu\left(\lambda^2 + 2\lambda^{-1} - 3\right),$$

and the stress-stretch relation is

$$s(\lambda) = \mu\left(\lambda - \lambda^{-2}\right).$$

The stress-stretch relation is an increasing function. Consequently, for a given s, the above equation determines a single stretch. The function $W(\lambda)$ is convex, so that every state of equilibrium is stable against small perturbation in stretch.

For a power-law material, the free energy function is



$$W(\lambda) = \frac{K}{N+1}(\log\lambda)^{N+1},$$

and the stress-stretch relation is

$$s(\lambda) = K\frac{(\log\lambda)^N}{\lambda}.$$

As the stretch increases from 1, the stress goes up, reaches a peak, and then drops. When the prescribed stress is above the peak, no state of equilibrium exists. When the prescribed stress is below the peak, two states of equilibrium exist. Of the two states of equilibrium, the one with smaller stretch is stable, but the one with larger stretch is unstable. This statement can be verified by checking the sign of $d^2W(\lambda)/d\lambda^2$. Part of the function $W(\lambda)$ is convex, and the other part is concave. The inflection point on the $W(\lambda)$ curve corresponds to the peak on the $s(\lambda)$ curve.

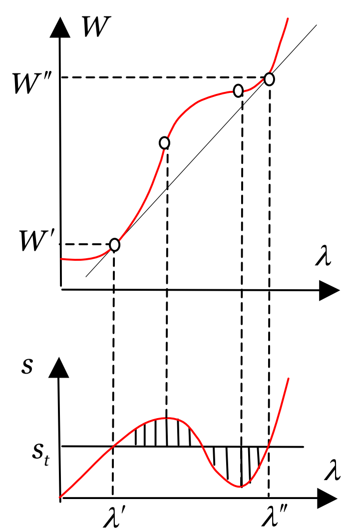
Exercise. For the nonlinearly elastic material obeying the power law, $\sigma = K\varepsilon^N$, determine the critical force P_c. Plot the function $\Pi(\lambda)$ for a value of P larger than P_c. Plot the function $\Pi(\lambda)$ for a value of P smaller than P_c. Comment on these plots.

Exercise. A spherical balloon is of radius R and thickness H in the undeformed state. Subject to a constant pressure p, the balloon expands to radius r. The balloon and the constant pressure together are regarded as a composite system. Show that the Helmholtz free energy of the composite system is

$$\Pi = 4\pi R^2 HW - \frac{4\pi}{3} r^3 p .$$

The free energy is a function of the radius of the balloon, $\Pi(r)$. Use this function to derive the condition of equilibrium and the condition for stability.

Coexistent phases. Maxwell's rule. Examples include coexistent martensite and austenite of a crystal, and coexistent swollen and collapsed phases of a gel. Such a phenomenon may be modeled by a Helmholtz free energy function $W(\lambda)$ of a shape shown in the figure. For such a $W(\lambda)$, the function $s(\lambda)$ is not monotonic.



In the reference state, Let L be the total length of the bar, and L' be the length of one phase, and L'' be the length of the other phase, so that

$$L = L' + L'' .$$

Let the stretches of the two phases be λ' and λ''. In the current state, the total length of the bar is the sum of the lengths of the two phases:

$$l = \lambda' L' + \lambda'' L'' .$$

The bar is subject to an axial force of a fixed magnitude *P*. The bar and fixed force together are regarded as a composite thermodynamic system. In the current state, the Helmholtz free energy of the composite is

$$\Pi = W(\lambda')AL' + W(\lambda'')AL'' - P(\lambda'L' + \lambda''L'').$$

This free energy is a function of three independent variables, $\Pi(\lambda', \lambda'', L')$. According to thermodynamics, the two phases equilibrate when the free energy $\Pi(\lambda', \lambda'', L')$ is minimized.

Associated with small changes of the three independent variables, the free energy changes by

$$\begin{aligned}\delta\Pi = &\frac{\partial W(\lambda')}{\partial \lambda'}AL'\delta\lambda' + W(\lambda')A\delta L' \\ &+ \frac{\partial W(\lambda'')}{\partial \lambda''}AL''\delta\lambda'' - W(\lambda'')A\delta L' \\ &- P(L'\delta\lambda' + \lambda'\delta L' + L''\delta\lambda'' - \lambda''\delta L')\end{aligned}$$

In equilibrium $\delta\Pi = 0$ for the three independent small changes $\delta\lambda', \delta\lambda'', \delta L'$. This condition of equilibrium leads to three independent equations:

$$s = \frac{\partial W(\lambda')}{\partial \lambda'},$$

$$s = \frac{\partial W(\lambda'')}{\partial \lambda''},$$

$$W(\lambda') - s\lambda' = W(\lambda'') - s\lambda''.$$

Here $s = P/A$ is the nominal stress. When the two phases coexist, the nominal stress in the two phases is the same. The first two equations are familiar. The third equation may be written as

$$s = \frac{W(\lambda'') - W(\lambda')}{\lambda'' - \lambda'}.$$

Thus, when the two phases are in equilibrium, the nominal stress is given by is the slope of the common tangent line of the function $W(\lambda)$.

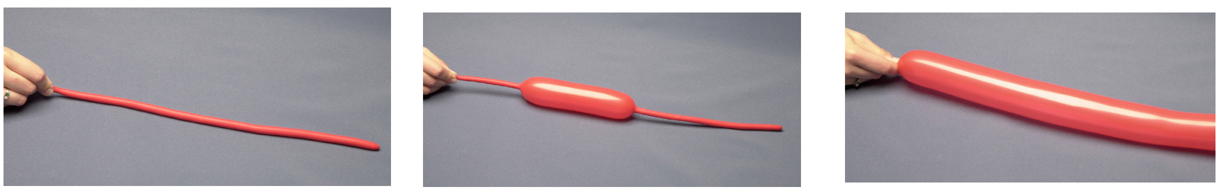
On the (λ, s) plane, the area under the curve $s(\lambda)$ represents the free energy $W(\lambda)$. Inspecting the graph, we can readily see that the above condition means that stress s for the two phases to equilibrate is at the level such that the two shaded areas are equal. This graphic interpretation is known as *Maxwell's rule*.

Also marked on the $W(\lambda)$ curve are the two inflection points, where

$$\frac{d^2W(\lambda)}{d\lambda^2} = 0.$$

The part of the $W(\lambda)$ curve between the inflection points is concave, where a single phase is unstable. Parts of the $W(\lambda)$ curve are convex, but are between the interval (λ', λ''). These parts correspond to metastable phases.

Exercise. Study the inflation of a cylindrical balloon. Can your solution interpret the observation of a party balloon?



http://www.doitpoms.ac.uk/tlplib/bioelasticity/demo.php

Reading. Chater, E., Hutchinson, J.W., On the Propagation of Bulges and Buckles. Journal of Applied Mechanics, 51, 1-9 (1984).
http://www.seas.harvard.edu/hutchinson/papers/377.pdf

T.Q. Lu and Z.G. Suo Large conversion of energy in dielectric elastomers by electromechanical phase transition. Acta Mechanica Sinica 28, 1106-1114 (2012). (http://www.seas.harvard.edu/suo/papers/285.pdf).

A cube under a hydrostatic stress revisited. If the hydrostatic stress is fixed as a true stress, the undeformed state is stable against deformation of any type and any magnitude, so long as no cavity forms in the material. This conclusion is understood as follows. Because the material is incompressible, when no cavity forms in the material, the volume of the body remains unchanged as it deforms. Consequently, the hydrostatic stress does no work as the body deforms, namely, the potential energy associated with the applied stress is zero. On the other hand, any deformation adds elastic energy. Consequently, any deformation of the block will increase the total Helmholtz free energy.

The situation is different if the cube is subject to equal-triaxial forces of a fixed magnitude *P*. The work done by the forces is

$$PL(\lambda_1 + \lambda_2 + \lambda_3 - 3) .$$

Recall the condition of incompressibility $\lambda_1\lambda_2\lambda_3 = 1$. The work is minimized at the trivial state $\lambda_1 = \lambda_2 = \lambda_3 = 1$. The work is positive when the force is tensile. When the force is large enough, this positive work will overcome the elastic energy associated with the deformation.

The block and the dead load together form a thermodynamic system. The free energy of the system is the sum of the Helmholtz free energy of the block and the potential energy of the dead load:

$$\Pi = L^3 W(\lambda_1, \lambda_2) - PL(\lambda_1 + \lambda_2 + \lambda_1^{-1}\lambda_2^{-1} - 3).$$

The free energy is a function of the two stretches, $\Pi(\lambda_1, \lambda_2)$. Thermodynamics requires that a stable state of equilibrium minimizes $\Pi(\lambda_1, \lambda_2)$. We assume that $W(1,1) = 0$, and $W(\lambda_1, \lambda_2) > 0$ when the cube deforms.

When the cube is undeformed, $\lambda_1 = \lambda_2 = 1$, the function $\Pi(\lambda_1, \lambda_2)$ vanishes. When the cube deforms, the elastic energy W increases Π, but the potential energy due to the dead load always reduces Π, provided the forces are tensile. Consequently, when the applied force P is tensile and is of a large enough magnitude, the undeformed state will no longer minimize the function $\Pi(\lambda_1, \lambda_2)$, and the cube will deform.

Pursuing this matter further, we examine the Taylor expansion:

$$\Pi(\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2) - \Pi(\lambda_1, \lambda_2) = \frac{\partial \Pi(\lambda_1, \lambda_2)}{\partial \lambda_1}\delta\lambda_1 + \frac{\partial \Pi(\lambda_1, \lambda_2)}{\partial \lambda_2}\delta\lambda_2$$
$$+ \frac{\partial^2 \Pi(\lambda_1, \lambda_2)}{2\partial \lambda_1^2}(\delta\lambda_1)^2 + \frac{\partial^2 \Pi(\lambda_1, \lambda_2)}{2\partial \lambda_2^2}(\delta\lambda_2)^2 + \frac{\partial^2 \Pi(\lambda_1, \lambda_2)}{\partial \lambda_1 \partial \lambda_2}(\delta\lambda_1 \delta\lambda_2)$$

Setting the first derivatives to zero, we obtain that

$$\frac{\partial \Pi(\lambda_1, \lambda_2)}{\partial \lambda_1} = 0, \quad \frac{\partial \Pi(\lambda_1, \lambda_2)}{\partial \lambda_2} = 0.$$

This is a pair of nonlinear algebraic equations that determine the two stretches of any state of equilibrium. To determine the stretches in equilibrium, you will need to specify a material model, $W(\lambda_1, \lambda_2)$, and a level of the dead load P. Multiple deformed states of equilibrium may exist.

For a state of equilibrium to be stable against the infinitesimal perturbation, the quadratic form

$$\frac{\partial^2 \Pi(\lambda_1, \lambda_2)}{2\partial \lambda_1^2}(\delta\lambda_1)^2 + \frac{\partial^2 \Pi(\lambda_1, \lambda_2)}{2\partial \lambda_2^2}(\delta\lambda_2)^2 + \frac{\partial^2 \Pi(\lambda_1, \lambda_2)}{\partial \lambda_1 \partial \lambda_2}(\delta\lambda_1 \delta\lambda_2)$$

must be positive-definite for any infinitesimal change in the stretches, $\delta\lambda_1$ and $\delta\lambda_2$. This condition, according to a theorem in linear algebra, requires that the following three conditions be satisfied:

$$\frac{\partial^2 \Pi(\lambda_1,\lambda_2)}{2\partial\lambda_1^2} > 0$$

$$\frac{\partial^2 \Pi(\lambda_1,\lambda_2)}{2\partial\lambda_2^2} > 0$$

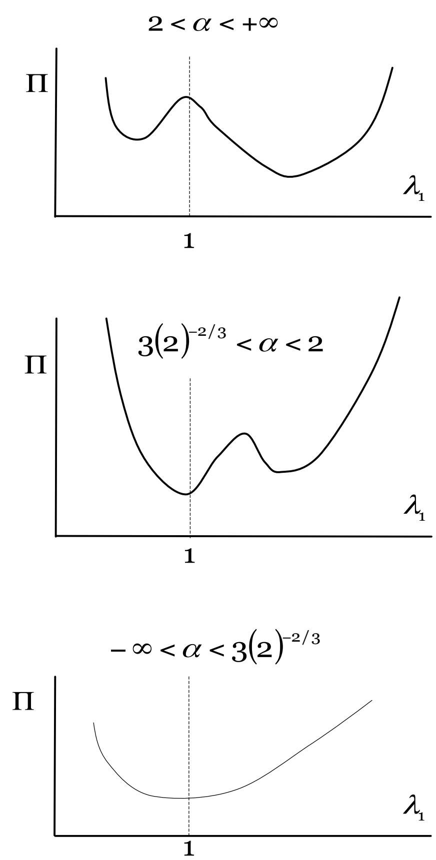
$$\left[\frac{\partial^2 \Pi(\lambda_1,\lambda_2)}{\partial\lambda_1^2}\right]\left[\frac{\partial^2 \Pi(\lambda_1,\lambda_2)}{\partial\lambda_2^2}\right] > \left[\frac{\partial^2 \Pi(\lambda_1,\lambda_2)}{\partial\lambda_1\partial\lambda_2}\right]^2$$

These conditions need be verified for each of the state of equilibrium to confirm its stability.

To check the stability of the equilibria, we plot the free energy function. For the neo-Hookean material, the Helmholtz free energy is

$$W(\lambda_1,\lambda_2) = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3\right).$$

We only need to consider the case $\lambda_1 = \lambda_2$, so that the free energy is a function of a single state variable:



$$\Pi(\lambda_1) = L^3 \frac{\mu}{2}\left(2\lambda_1^2 + \lambda_1^{-4} - 3\right) - PL\left(2\lambda_1 + \lambda_1^{-2} - 3\right)$$

The shape of the function changes with the parameter $P/\left(L^2\mu\right)$, which is designated as α in the figure.

Exercise. Modify the above model to study the effect of compressibility. Write the Helmholtz free energy as

$$W(\lambda_1,\lambda_2) = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + \frac{K}{2}\left(\lambda_1\lambda_2\lambda_3 - 1\right)^2.$$

Exercise. Modify the above model for a sheet of a rubber subject to equal biaxial forces. Show that the stretches in two directions in the plane of sheet can be unequal. (E.A. Kearsley, Asymmetric stretching of a symmetrically loaded elastic sheet. Int. J. Solids Structures 22, 111-119, 1986)

Other modes of instability. Of all possible states, the stable state of equilibrium minimizes the free energy. This statement is a basic principle of thermodynamics. But the principle can be hard to implement: it is hard to seek a minimizer among **all** possible states. Say we try to find the tallest human being ever lived. It will not do if we just look among all living Americans. We have to search world-wide. Should we extend our search beyond the Earth? Should we go back to history to look among dead people? Then we do not have good records of people's heights. We will also meet the controversy at what point we wish to call something human.

Thus, in searching for **the state** that minimizes the free energy, we will have to be clear what we mean by **all possible states**. For the bar, we have restricted our search among all states of homogenous deformation. We have excluded states of inhomogeneous deformation. We have formulated a problem solvable in the classroom, but have lost our contact with the experimental observation of necking, which obviously involves inhomogeneous deformation.

If we broaden our search, and include other states as candidates, will we find other states of equilibrium? Yes. Here are some of other models of instability.

Buckle. When we restrict the states to be homogenous, the homogenous state is stable when the force is compressive. When we broaden the search to include bending states, the bar buckles when the compressive force reaches a critical value.

Wrinkle. For bar not very long compared to its width, the homogenous state can be unstable against perturbation of the shape of the surface, even when the compressive force is below the critical load of Euler buckling. M. A. Biot, Surface instability of rubber in compression, *Appl. Sci. Res. A* 12, 168 (1963).

Crease. In all above analysis, the search has been restricted among fields of displacement whose amplitude only slightly deviates from the homogeneous state. Experiments have shown that a block of rubber form creases at even a lower compressive forces. A. N. Gent, I. S. Cho, Surface instabilities in compressed or bent rubber blocks, *Rubber Chemistry and Technology* 72, 253 (1999). This mode of instability

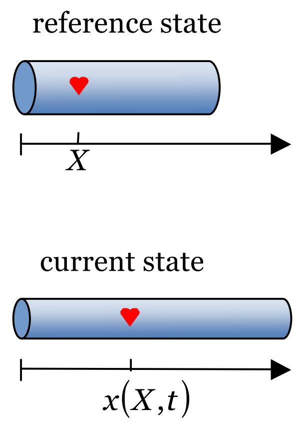
has been analyzed recently by E. Hohlfeld and L. Mahadevan (Phys. Rev. Letts. 106, 105702, 2011) In creasing, the amplitude of the field deviates greatly from the homogenous state, even though the spatial extent of an initial crease can be small. This mode of instability can also be analyzed by minimizing the free energy (Hong, Zhao, Suo, Applied Physics Letters 95, 111901, 2009, http://www.seas.harvard.edu/suo/papers/221.pdf).

Rupture. A rubber band in tension does not form a neck, but does rupture when the stretch is very large.

Vaporization. If we wait long enough, all solids will oxidize, vaporize, or further decompose to more elementary particles. Do we wish to include such states of matter in our search for the energy minimizer?

Inhomogeneous, Time-dependent Deformation

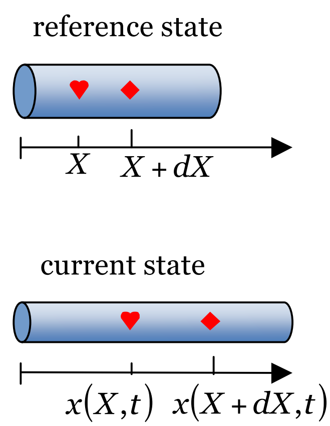
Dynamics of a bar. Subject to loads, a body in general undergoes inhomogeneous, time-dependent deformation. The general formulation is given in the notes on general theory of finite deformation (http://imechanica.org/node/538). Here we consider inhomogeneous and time-dependent deformation of a bar. We next list the ingredients of the theory.



Kinematics. The bar in the undeformed state is taken to be the reference state. The bar is modeled as a field of material particles. Name each material particle by the coordinate X of its place when the bar is in the reference state. In the current state t, the material particle X moves to a place of coordinate x. The kinematic of the deformation of the bar is described by the time-dependent field

$$x = x(X,t).$$

Evolving this field is the aim of the theory.



Consider a differential element of the bar. In the reference state, the element is between two material particles, X and $X + dX$. In the current state at time t, the material particle X moves to place $x(X,t)$, and the material particle $X + dX$ moves to place $x(X + dX,t)$. Recall that the stretch is defined as

$$\lambda = \frac{\text{length in current state}}{\text{length in reference state}}.$$

Thus, the stretch of the differential element is

$$\lambda = \frac{x(X + \mathrm{d}X, t) - x(X, t)}{\mathrm{d}X},$$

or

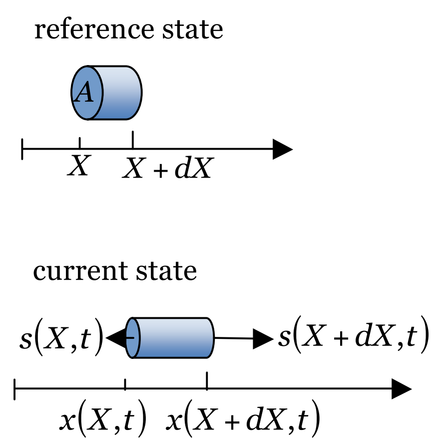
$$\lambda = \frac{\partial x(X, t)}{\partial X}.$$

The stretch is also a time-dependent field, $\lambda(X, t)$.

Conservation of mass. Define the nominal mass density by the mass in the current state divided by the volume in the reference state, namely,

$$\rho = \frac{\text{mass in the current state}}{\text{volume in the reference state}}.$$

We assume that that no mass transfers from one material particle to another. The conservation of mass dictates that the nominal mass density be time-independent. The density, however, may vary from one material particle to another. We may write the nominal mass density as a function of material particles, $\rho(X)$. In the following analysis, we assume that the bar is uniform in the reference state, so that the density is constant independent of time and particle.



Newton's second law. The nominal stress in the bar also varies from one material particle to another, and from one time to another. Consequently, the nominal stress in the bar is a time-dependent field, $s(X, t)$. When the bar is in the reference state, its cross-sectional area is A. Draw a free-body diagram of a differential element of the bar, between particles X and $X + dX$. The mass of the element is $\rho A dX$. The acceleration of the element is $\partial^2 x(X, t) / \partial t^2$. The net force applied on the element is $[s(X + dX, t) - s(X, t)]A$. Applying Newton's second law to this element at the current state at time t, we obtain that

$$[s(X + dX, t) - s(X, t)]A = (\rho A dX)\frac{\partial^2 x(X, t)}{\partial t^2}.$$

This equation is re-written as

$$\frac{\partial s(X,t)}{\partial X} = \rho \frac{\partial^2 x(X,t)}{\partial t^2}.$$

Material model. The material is taken to be nonlinearly elastic with the stress-stretch relation

$$s = g(\lambda).$$

This function is measured by applying to a short segment of the bar a uniaxial force, which causes a state of homogeneous deformation. We assume that the same function is valid locally for inhomogeneous deformation of the bar.

Summary of the three ingredients. The bar is characterized by three time-dependent fields: deformation $x(X,t)$, stretch $\lambda(X,t)$, and nominal stress $s(X,t)$. The stretch is defined by the gradient of the deformation:

$$\lambda = \frac{\partial x(X,t)}{\partial X}.$$

Newton's law relates the fields of stress and deformation:

$$\frac{\partial s(X,t)}{\partial X} = \rho \frac{\partial^2 x(X,t)}{\partial t^2}.$$

A material model is specified by a stress-stretch relation:

$$s = g(\lambda).$$

The two partial differential equations are linear. The material model is an algebraic relation between the stress and the stretch, which in general is nonlinear. A class of solutions involves linearization of the material model, so that all equations are linear. An example follows.

Wave in a pre-stressed bar. Linear perturbation. We now mix the three ingredients for a specific a phenomenon. A bar has a uniform density and a uniform cross-sectional area. The bar is in a state of homogenous deformation of stretch λ_0. We then launch into the bar a longitudinal wave of small amplitude. What will be the speed of the wave?

This example will illustrate a method known as *linear perturbation*. We begin with a known state of finite deformation. In this example, this state is a bar undergoing a static and homogenous deformation. Superimposed on the homogenous deformation is an inhomogeneous, time-dependent field of displacement $u(X,t)$. In the current state *t*, the material particle *X* moves to a place with the coordinate

$$x(X,t) = \lambda_0 X + u(X,t).$$

Taking derivative with respect to *X*, we obtain the stretch in the bar:

$$\lambda = \lambda_0 + \frac{\partial u(X,t)}{\partial X}.$$

The stretch in the bar is a time-dependent and inhomogeneous field, given by the sum of the homogenous stretch due to the pre-stress and an inhomogeneous strain due to the disturbance.

In the linear-perturbation analysis, the unperturbed state can be a state of finite deformation, but the perturbation in strain is assumed to be small, namely,

$$\left| \frac{\partial u(X,t)}{\partial X} \right| << 1.$$

This assumption enables us to use the Taylor expansion.

The material is taken to be nonlinearly elastic, characterized by a function between the stress and the stretch $s = g(\lambda)$. We expand the function $s = f(\lambda)$ into the Taylor series around λ_0:

$$s(\lambda) = g(\lambda_0) + g'(\lambda_0)(\lambda - \lambda_0).$$

Because the deviation from the homogeneous stretch, $\lambda - \lambda_0$, is small, the expansion is carried to the term linear in the deviation. The first derivative

$$g'(\lambda_0) = \left. \frac{ds(\lambda)}{d\lambda} \right|_{\lambda=\lambda_0}$$

is the tangent stiffness.

A mix of the three ingredients leads to the equation of motion:

$$g'(\lambda_0) \frac{\partial^2 u(X,t)}{\partial X^2} = \rho \frac{\partial^2 u(X,t)}{\partial t^2}.$$

This is a familiar linear partial differential equation, the wave equation. The equation evolves the disturbance $u(X,t)$. Consequently, the disturbance propagates in the bar at the speed

$$c = \sqrt{\frac{g'(\lambda_0)}{\rho}}.$$

The wave slows down when the tangent modulus reduces.

The wave takes the form

$$u(X,t) = F(X - ct) + G(X + ct).$$

Any arbitrary functions *F* and *G* satisfy the equation of motion. The forms of the two functions are determined by the initial and boundary conditions.

Considère condition reconsidered. The above analysis is sensible only when the tangent modulus is positive, $g'(\lambda_0) > 0$. When the tangent modulus vanishes, $g'(\lambda_0) = 0$, a time-independent inhomogeneous deformation becomes possible. Observe that the condition of vanishing tangent modulus is the same as the Considère condition.

In the previous treatment of necking, we restricted ourselves to homogeneous and time-independent fields. We found that the nominal stress reaches a peak at a certain stretch. We then jumped to the conclusion, asserting that the peak of the $s(\lambda)$ curve corresponds to the onset of necking. That model of necking was unsatisfactory. The model did not show the most salient feature of necking: an inhomogeneous deformation. The model could not continue the calculation after the peak.

Now we have a model of inhomogeneous and time-dependent deformation. Given a boundary condition and an initial condition, the model evolves the field of deformation $x(X,t)$. In particular, we can start with an initial condition of an inhomogeneous deformation. The magnitude of the inhomogeneity can be kept small to represent the imperfection of the bar. We then evolve the field in time, and see if the inhomogeneity amplifies. We can also vary the rate of pulling, and study the effect of inertia.

Of course, we have to be careful about any conclusion we draw from such a model. After all, necking is a three-dimensional process, and our model is one-dimensional. For example, the solution to the one-dimensional model is suspect if inhomogeneity occurs over a length shorter than the diameter of the bar.

Exercise. **Small-amplitude oscillation around a state of equilibrium of finite deformation**. Formulate the problem. What happens when the frequency of the fundamental mode approaches zero?

A Column Subject to a Compressive Axial Force

The connection between instability and dynamics also arises in another pair of familiar phenomena: vibration and buckling. Demonstrate in class the effect of axial force on frequency in class using a column with a variable length.

The column is simply supported at the two ends, and is subject to an axial force P in compression. The trivial solution is that the column remains straight under compression. We are looking for non-trivial solutions, in which the column bends.

Bending rigidity. You have studied the theory of beams in a previous course. Here is a summary of the key ideas: the three elements of mechanics applied to a small segment of a beam.

Geometry of bending. When a beam is bent, one side of the beam is in tension, and other side is in compression. A line in the cross section of the beam is under no strain. This unstrained line is called the neutral axis. For a cross section of a symmetric shape, such as a circle or a rectangle, the neutral axis is the line of symmetry.

Let z be an axis pointing to the tensile side of the beam, and let the origin of the axis lie on the neutral axis. It is assumed that, during deformation, the plane of the cross section rotates but remains planar. Let the radius of curvature by r, and the curvature be $\kappa = 1/r$. The strain in the beam in the axial direction is inhomogeneous, and is given by

$$\varepsilon = \kappa z .$$

The curvature is constant within the cross section, and the distribution of the strain in the cross section is linear in z. This distribution results from the assumption that the plane remains planar during deformation.

Balance of moments. The axial stress in the beam is also inhomogeneous, written as function $\sigma(z)$. Let M be the moment that bends the beam. The balance of moments requires that

$$M = \int z\sigma(z)dA .$$

The integral extends over the area of the cross section.

Material model. The strain in the column is small. Here we assume that the column remains elastic, and the stress-strain relation is given by Hooke's law:

$$\sigma = E\varepsilon ,$$

where E is Young' modulus of the material.

Mixing the three ingredients. A combination of the three ingredients relates the moment to the curvature:

$$M = EI\kappa ,$$

where

$$I = \int z^2 dA$$

is a geometric property of the cross section, known as the second moment of area. The second moment of area is $I = BH^3/12$ for a rectangle of height H and width B, and is $I = \pi R^4/4$ for a circle of radius R. The product EI is known as the bending rigidity.

Geometry of a bending. In the reference state, the column is straight and is of length L. Label material particles by the coordinate X in the reference state. In the current state at t, the column has a small deflection y. The function

$$y = y(X,t)$$

describes the bending movement of the column. The deflection is assumed to be small, so that formulas in the differential geometry of curves take simplified forms. The slope of the column is

$$\theta = \frac{\partial y(X,t)}{\partial X}.$$

The curvature of the column is

$$\kappa = -\frac{\partial^2 y(X,t)}{\partial X^2}.$$

Buckling. We first look for a state of equilibrium. The shear force in the beam vanishes. We balance the moment in the current state:

$$M = Py.$$

(We will miss this term if we balance the moment in the reference configuration.)

Recall the moment relates to the curvature by $M = EI\kappa$, and the curvature relates to the deflection by $\kappa = -\partial^2 y / \partial X^2$. The equation $M = Py$ becomes

$$EI\frac{\partial^2 y}{\partial X^2} + Py = 0.$$

The function $y = \sin(\pi X / L)$ satisfies the equilibrium equation and the boundary conditions, provided that the axial force is given by

$$P_c = \pi^2 \frac{EI}{L^2}.$$

This is the Euler condition for bucking.

Vibration. We next look for a dynamic solution. The shear force is present, denoted as $Q(X,t)$. Consider an element of the beam from X to $X + dX$. Balancing the moment of the element, we obtain that

$$\frac{\partial M(X,t)}{\partial X} = P\frac{\partial y(X,t)}{\partial X} + Q.$$

Applying Newton's second law in the vertical direction, we obtain that

$$\frac{\partial Q(X,t)}{\partial X} = \rho A\frac{\partial^2 y(X,t)}{\partial t^2},$$

where ρ is the mass density and A the area of the cross section.

Recall the moment relates to the curvature by $M = EI\kappa$, and the curvature relates to the deflection by $\kappa = -\partial^2 y / \partial X^2$. A combination of these equations gives that

$$-EI\frac{\partial^4 y}{\partial X^4} - P\frac{\partial^2 y}{\partial X^2} = \rho A\frac{\partial^2 y}{\partial t^2}.$$

This expression is the equation of motion for the deflection $y(X,t)$.

Try the solution of the form

$$y(X,t) = \sin\left(\frac{\pi X}{L}\right)\sin\omega t.$$

This form satisfies the boundary conditions of simply supported ends. This form also satisfies the equation of motion, provided the frequency is

$$\omega = \pi^2\sqrt{\frac{EI}{\rho AL^4}\left(1 - \frac{P}{P_c}\right)}.$$

The frequency decreases when the compressive force increases, and vanishes when the axial force approaches the critical load for buckling.

Wave. The equation of motion is

$$-EI\frac{\partial^4 y}{\partial X^4} - P\frac{\partial^2 y}{\partial X^2} = \rho A\frac{\partial^2 y}{\partial t^2}.$$

Look for solution of the form

$$y(X,t) = f(X)\sin(\omega t).$$

Inserting this expression into the equation of motion, we obtain that

$$EI\frac{d^4 f}{dX^4} + P\frac{d^2 f}{dX^2} = A\rho\omega^2 f.$$

This is an ordinary differential equation for $f(X)$. The bounded solution is a linear combination of $\sin(kX)$ or $\cos(kX)$, where the wavenumber k is given by

$$k = \left(\frac{P + \sqrt{P^2 + 4EI\rho A\omega^2}}{2EI}\right)^{1/2}.$$

The wavenumber is a nonlinear function of the frequency, so that the wave in the beam is dispersive.