Fracture of Rubber. Lecture 2

Fracture mechanics without invoking any field theory. In Lecture 1 on Fracture of Rubber, we considered the extension of a crack in an elastic body subject to a load. Following Rivlin and Thomas (1953), we regarded the elastic energy stored in the body as a function of two independent variables: the displacement of the load, and the area of the crack. The partial derivative of the elastic energy with respect to the area of the crack defined the energy release rate (http://imechanica.org/node/7773).

This definition of the energy release rate invokes no field theory. Indeed, the energy release rate can be determined experimentally without measuring any field. The energy release rate has been used as a loading parameter to study diverse phenomena related to the extension of a crack. Examples include fracture energy, *R*-curve, fatigue, and stress corrosion. See Lake (2003) for a review of these studies for rubber.

All these applications of fracture mechanics require no field theory. Fracture mechanics applies whether the material is glass or rubber, and whether deformation is small or large. Rivlin and Thomas (1953) attributed this approach to Griffith (1921).

The condition of small-scale fracture process zone. In defining the energy release rate, we have assumed that the body is elastic. In reality, the extension of a crack in the body is always an inelastic process, involving breaking atomic bonds, growing cavities, etc. For a material such as glass or rubber, our daily experience suggests that the inelastic process of fracture is mostly confined in zone around the front of the crack.

So long as this fracture process zone is small compared to the size of a specimen (e.g., the length of the crack, the length of the ligament), the shape of the specimen and the distribution of the load should not affect the fracture process. The magnitude of the load is represented by the energy release rate G, which is the only parameter that links the external mechanical boundary conditions and the fracture process.

What do we gain by specifying a field theory in fracture mechanics? Most engineering materials, however, can be modeled by field theories of one kind or another. For example, Griffith (1921) modeled glass using the linear elastic theory. What do we gain by introducing such a field theory into fracture mechanics? Here are two basic consequences:

1. The energy release rate G is quadratic in applied load. For a given cracked body, G can be determined by solving a boundary-value problem. http://imechanica.org/node/7507
2. The crack-tip field is square-root singular. The intensity of the field is specified by G. http://imechanica.org/node/7579

Consequence 1 is of immediate practical value. Compared to determining G by experimental measurement, it is often convenient to calculate G using the finite element method. Consequence 2 helps to quantify the condition of small-scale fracture process zone.

Both consequences have analogs when a material is modeled with the nonlinear elastic theory. This lecture reviews the nonlinear elastic theory, and describes its consequences for fracture of rubber.

Nonlinear elastic theory. We recall the nonlinear elastic theory. The theory consists of the usual ingredients: geometry of deformation, material model, and condition of equilibrium (http://imechanica.org/node/5065). These ingredients are reviewed here.

Geometry of deformation. An elastic body is represented by a field of material particles. Each particle is named after its coordinate $\mathbf{X}$ when the body is in a state of reference. When the body is in a deformed state, the particle $\mathbf{X}$ is at a place of coordinate $\mathbf{x}$. The deformation of the entire body is described by the function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}).$$

Define the deformation gradient by

$$F_{iK} = \frac{\partial x_i(\mathbf{X})}{\partial X_K}.$$

When the particles move by $\delta x_i(\mathbf{X})$, the deformation gradient varies by

$$\delta F_{iK} = \frac{\partial}{\partial X_K}\delta x_i(\mathbf{X}).$$

Material model. Let W be the elastic energy in the deformed state per unit volume of the state of reference. The elastic energy stored in the body is

$$U = \int W dV.$$

The integral is carried over the volume of the body in the state of reference.

Specify an elastic material by prescribing the energy density W as a function of the deformation gradient, namely,

$$W = W(\mathbf{F}).$$

When the particles move by $\delta x_i(\mathbf{X})$, the energy density varies by

$$\delta W = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}\delta F_{iK}.$$

The partial derivative defines the nominal stress:

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}.$$

The equation provides the stress-stretch relation once the energy-density function is prescribed.

Conditions of equilibrium. One part of the surface of the body is prescribed with displacement, and the other part is prescribed with traction **T**. The traction does work

$$\int T_i \delta x_i dA.$$

The integral extends over the part of the surface of the body where the traction is applied. In equilibrium, the variation in the elastic energy of the body equals the work done by the traction, namely,

$$\delta U = \int T_i \delta x_i dA.$$

The equality holds for arbitrary variation of the position. A combination of the above equations leads to the conditions of equilibrium:

$$\frac{\partial s_{iK}(\mathbf{X})}{\partial X_K} = 0$$

inside the body, and

$$s_{iK} N_K = T_i$$

on the surface of the body where the traction is applied. Here $\mathbf{N}$ is the unit vector normal to the surface when the body is in the undeformed state.

Neo-Hookean model. Subject to a load, rubber changes shape more readily than it changes volume. Consequently, rubber is usually assumed to be incompressible, namely,

$$\det\mathbf{F} = 1.$$

The neo-Hookean model is specified by the energy density function:

$$W = \frac{\mu}{2} F_{iK} F_{iK},$$

where μ is the shear modulus.

Arruda-Boyce model. The neo-Hookean model misses an essential feature of rubber: the stress-stretch stiffens sharply when polymer chains approach their contour length. The

Arruda-Boyce model accounts for this feature. In this model, rubber is taken to be incompressible:

$$\det \mathbf{F} = 1.$$

The network is represented by the eight-chain model, so that the stretch of each chain is

$$\Lambda = \left(\frac{F_{iK}F_{iK}}{3} \right)^{1/2}.$$

Each chain is modeled as a chain of freely jointed links. For each chain, the stretch Λ relates to the dimensionless force ζ as

$$\Lambda = \sqrt{n}\left(\frac{1}{\tanh \zeta} - \frac{1}{\zeta} \right),$$

where n is the number of links in the chain.

The free energy per unit volume of the elastomer is

$$W = \frac{kT}{v}\left(\frac{\zeta}{\tanh \zeta} - 1 + \log \frac{\zeta}{\sinh \zeta} \right),$$

where kT is the temperature in the unit of energy, and v the volume per link.

The above equations define the energy-density function $W(\mathbf{F})$ through two intermediate parameters: the stretch Λ and the normalized force ζ in each chain. In one limit, $\zeta \to \infty$, the chain approaches the limiting stretch, $\Lambda \to \sqrt{n}$. In the other limit, $\zeta \to 0$, the chain coils much below the limiting stretch, $\Lambda << \sqrt{n}$, and the model reduces to $W = (kT/6v)\zeta^2$ and $\Lambda = (\sqrt{n}/3)\zeta$, which recovers the neo-Hookean model, with the shear modulus $\mu = kT/(vn)$.

Dimensional analysis of boundary-value problems. For a crack in an elastic body subject to a load, a boundary-value problem is specified by

- Energy density function
- Boundary conditions

A few such boundary-value problems have been solved analytically. In general, boundary-value problems can be solved by finite element method. The commercial code ABAQUS also calculates the energy release rate.

Here we describe the structure of solutions by dimensional analysis. Let μ be the representative value of modulus. We write the energy density function as

$$W = \mu \hat{W}(\mathbf{F}, \alpha).$$

We will use $\hat{f}(\ )$ to represent a dimensionless function of dimensionless variables. Here α represents dimensionless parameters that specify a material model. For example, the number of links in a freely-jointed chain is such a parameter.

The boundary conditions are specified by a representative length L in the body in the state of reference, by a representative stretch λ as a loading parameter, and by a set of dimensionless parameters to describe aspect ratios, etc.

The solution of the boundary-value problem takes the following form. The body deforms to

$$x_i = L\hat{x}_i\left(\frac{\mathbf{X}}{L}, \lambda, \alpha, \beta \right).$$

The deformation gradient takes the form

$$F_{iK} = \hat{F}_{iK}\left(\frac{\mathbf{X}}{L}, \lambda, \alpha, \beta \right).$$

The stress takes the form.

$$s_{iK} = \mu \hat{s}_{iK}\left(\frac{\mathbf{X}}{L}, \lambda, \alpha, \beta\right).$$

The energy release rate takes the form:

$$G = L\mu\hat{G}(\lambda, \alpha, \beta).$$

Examples of boundary-value problems. For a crack of a length 2*a* in an infinite body, of a Neo-Hookean material, subject to stretch remote stress λ, the energy release rate takes the form

$$G = a\mu\hat{G}(\lambda).$$

For such a crack in a body of the Arruda-Boyce material, the energy release rate takes the form

$$G = a\frac{kT}{v}\hat{G}(\lambda, n).$$

The energy release rate is expected to decrease when *n* increases.

For solutions of boundary-value problems, see a review by Hocine and Abdelaziz (2009).

Dimensional analysis of crack-tip field. Consider a semi-infinite crack in an infinite body of a material characterized by a modulus μ and a dimensionless parameter α. The size of the body and the distribution of the load enter the problem through the energy release rate *G*. This problem has a length scale:

$$G/\mu.$$

The field takes the form

$$x_i = \frac{G}{\mu}\hat{x}_i\left(\frac{\mathbf{X}}{G/\mu}, \alpha\right),$$

$$F_{iK} = \hat{F}_{iK}\left(\frac{\mathbf{X}}{G/\mu}, \alpha\right),$$

$$s_{iK} = \mu\hat{s}_{iK}\left(\frac{\mathbf{X}}{G/\mu}, \alpha\right).$$

Size of fracture process zone. The energy release rate is defined for an elastic body. Fracture is an inelastic process. The energy release rate can be used as a driving force for fracture, so long as the size of the fracture process zone, r_p, is small compared to the representative size of the specimen (e.g., length of the crack, length of the ligament).

How large is the fracture process zone? Suppose that our choice of the elastic model,

$$W = \mu\hat{W}(\mathbf{F}, \alpha),$$

ceases to be accurate when the stretch reaches order unity, or the stress reaches the order of the modulus. We lump the region that exceeds this level of stretch and stress into the fracture process zone. Within this description, the fracture process zone is estimated by

$$r_p \sim \frac{\Gamma}{\mu},$$

where Γ is the fracture energy, and μ is the shear modulus. For representative values $\Gamma = 10^3 \mathrm{J/m^2}$ and $\mu = 10^6 \mathrm{J/m^3}$, this estimate gives $r_p = 1\mathrm{mm}$.

Singular field around the tip of a crack in a neo-Hookean material. This problem has been solved analytically (e.g., Knowles and Sternberg, 1972). The field of deformation is

$$x_1 = CR\cos\Theta$$
$$x_2 = 2\sqrt{\frac{GR}{\pi\mu}}\sin\frac{\Theta}{2}$$

The leading-order singularity is governed by G. The dimensionless constant C is the horizontal stretch.

The nominal stresses are

$$s_{11} = \mu C$$
$$s_{12} = 0$$
$$s_{21} = -\mu\sqrt{\frac{G}{\pi\mu R}}\sin\frac{\Theta}{2}$$
$$s_{22} = \mu\sqrt{\frac{G}{\pi\mu R}}\cos\frac{\Theta}{2}$$

The true stresses are

$$\sigma_{11} = \mu C^2,$$
$$\sigma_{12} = \sigma_{21} = -\mu C\sqrt{\frac{2G}{\pi\mu R}}\sin\frac{\Theta}{2}$$
$$\sigma_{22} = \frac{G}{\pi R}$$

For a review of crack-tip field in rubber, see Krishnan et al. (2008).

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