The Griffith Paper

Required reading. A.A. Griffith, The phenomena of rupture and flow in solids. Philosophical Transactions of the Royal Society of London, A 221, 163-198 (1921). This paper initiated the theory of fracture, and has foreshadowed much of the subsequent development. Start reading the paper today, and return to it for illumination later in your career.

Qualitative Content of the Griffith Paper

The age of atomic picture for everything. After the atomic nature of matter was confirmed by many experimental observations, about a century ago, it became fashionable and useful to use atomic processes to understand macroscopic phenomena. In 1921 the British engineer A.A. Griffith published a paper on one such macroscopic phenomenon: fracture of a glass. The main puzzle had been that the glass usually breaks under a stress several orders of magnitude below the strength of atomic cohesion.

Griffith took up the notion that a body of glass is never perfect: flaws pre-exist in the body. A flaw with a sharp tip greatly concentrates stress. The intense stress at the tip of the flaw breaks atomic bonds one by one, like opening a zipper. The flaw turns into an advancing crack, and the body breaks into two halves.

This notion is easy to picture, but difficult to quantify. The essential difficulty is that the material behavior at the tip of the flaw is nonlinear. It is unclear what permitted Inglis (1913) to use the linear elastic theory to calculate the stress at the tip of the flaw. A direct approach to resolve this difficulty would be to use a nonlinear material model. For example, one may invoke atomistic simulations. Atomistic simulations of fracture have been pursued since the time of Inglis, but have not been widely used in engineering practice to this day.

The Griffith approach. Griffith took a less direct approach. Consider a pre-existing crack in a body subject to an external force. Regard the body and the external force together as a thermodynamic system. Characterize the system by two thermodynamic variables: the area of the crack and the displacement of the loading grips. To focus on essential ideas, suppose that, after a certain amount of displacement, the loading grips are held fixed, but the crack is allowed to advance. Because the loading grips are held fixed, the external force does no work. The energy of the system is the sum of the elastic energy in the body, and the surface energy in the faces of the crack. The energy of the system is a function of a single thermodynamic variable: the area of the crack. When the crack advances, the stress in the sample is partially relieved, so that the elastic energy is reduced. At the same time, the advancing crack creates more surface area, so that the surface energy increases. Thermodynamics dictates that the process should go in the direction that reduces the total free energy. If the decrease in elastic energy prevails, the crack grows. If the increase in surface energy prevails, the crack heals.

The nonlinear zone, localized around the tip of the crack, remains invariant as the crack advances. Consequently, the presence of the nonlinearity does not affect the accounting of the change in energy associated with the advance of the crack. The Griffith approach circumvents the nonlinear crack-tip behavior by invoking one quantity: the surface energy. The science of fracture was born.

To sum up, the qualitative content of the Griffith theory is

- In a body of a glass cracks pre-exist.
- The tip of such a crack concentrates stress.
- The intense stress breaks atomic bonds one by one, like opening a zipper.
- As the crack advances, fresh surfaces are created. The surface energy increases, but the elastic energy decreases.
- The crack advances if the advance reduces the sum of the surface energy and elastic energy.

I'll do two things in class. In formulating his theory, Griffith used this quantity surface energy. For some of us, the last time we encountered the surface energy was in kindergarten, when we blew soap bubbles. So I'll first outline aspects of the surface energy. I'll then describe the content of the Griffith paper.

Surface Energy

Surface energy. An atom at the surface of a body has a bonding environment different from that of an atom inside the body. The free energy per atom at the surface is higher than the free energy per atom in the body. The excess defines the surface energy.

The above description appeals to intuition, but is not operational. An atom does not have its private energy: the energy is in the bonds between atoms. What do we really mean by the phrase "free energy per atom"? For a bulk, this phrase simply means "the free energy of the bulk divided by the number of atoms in the bulk". But when we wish to compare atoms in the interior of the bulk and those on the surface, the meaning of the phrase "free energy per atom" becomes unclear, because we have not specified what we mean by "free energy on the surface".

An operational definition of the surface energy goes like this. Imagine a body whose size is much larger than an individual atom. Denote the energy of this body by U_o. It includes all the energy stored in electron clouds, or even the nuclear energy stored in the atomic nuclei. Next imagine that this body is split in two halves. The act of splitting may cause some damage of the material, e.g., introducing some dislocations or some microcracks. Let's say we are very careful in the act of splitting, and anneal the samples afterwards, so that such damage is eliminated. All atoms on the two surfaces relax to their equilibrium configurations. The energy in the two halves will be greater than U_o. Let's call the energy in the two halves U. The difference, $U - U_o$, is defined as the surface

energy. Of course, this difference does not belong to a single layer of atoms on the surface. It is a collective effect. For this definition to be useful, we take advantage of a physical fact: atoms a few layer beneath the surface recover the configuration of atoms in the bulk.

Let γ be the surface energy per unit area. The quantity γ is called the surface energy density, or surface energy for brevity. The surface energy density is independent of the size of the body, unless the body approaches the atomic dimension.

- A.W. Adamson, *Physical Chemistry of Surfaces*. Wiley, New York (1990). Chapter 1 describes many phenomena concerning the effect of surface energy for liquids.
- P.G. de Gennes, F. Brochard-Wyart, and D. Quere, Capillarity and Wetting Phenomena, Springer (2004).

Magnitudes of the surface energy. Surface energy scales with the cohesive energy of bonds in a substance. Here are some representative values.
Metals and ceramics: ~1 J/m²
Water: 0.07 J/m². Most other liquids have smaller values.
Polymers: 0.01-0.1 J/m²
Here are interpretations of the magnitudes of surface energy in the book by de Gennes et al.

When segregated to the surface, a liquid molecule is in an unfavorable energy state. If the cohesion energy per molecule is U inside the liquid, a molecule sitting at the surface finds itself short of roughly $U/2$. The surface tension is a direct measure of this energy shortfall per unit surface area. If a is the molecule's size and a^2 is its exposed area, the surface tension is of order $\gamma \cong U / \left(2a^2 \right)$. For most oils, for which the interactions are of the van der Waals type, we have $U \cong kT$, which is the thermal energy. At a temperature of $25^\circ \mathrm{C}$, kT is equal to 1/40 eV, which gives $\gamma \cong 20 \, \mathrm{mJ/m^2}$.

Because water involves hydrogen bonds, its surface tension is larger ($\gamma \approx 72 \, \mathrm{mJ/m^2}$). For mercury, which is a strongly cohesive liquid metal, $U \approx 1 \, \mathrm{eV}$ and $\gamma \approx 500 \, \mathrm{mJ/m^2}$. Note that γ can equivalently be expressed in units of mN/m.

Note that an elastomer is similar to a liquid at the molecular level. In the polymer network, each polymer chain consists of a large number of monomers. When the elastomer is stretched, molecules in the interior come to the surface. The surface tension of the elastomer should be similar to that of oil.

Measuring the surface energy of a liquid. A U-shaped rigid frame is fixed in space. A liquid membrane lies in the area confined by the frame and a slider. The slider can move without friction. The thickness of the membrane is much larger than the dimension of the individual molecule of the liquid. As the slider moves to the right, the membrane becomes thinner and has a larger area. Molecules are drawn from the interior of the membrane to the surfaces. A force F is applied on the slider. What is the force needed to maintain equilibrium?

You can always represent a constant force F by a hanging weight. The membrane and the weight together constitute a thermodynamic system. The free energy of the system is

$$U = 2\gamma bx - Fx .$$

Here $2\gamma bx$ is the surface energy of the two faces of the membrane, and $-Fx$ is the potential energy of the weight.
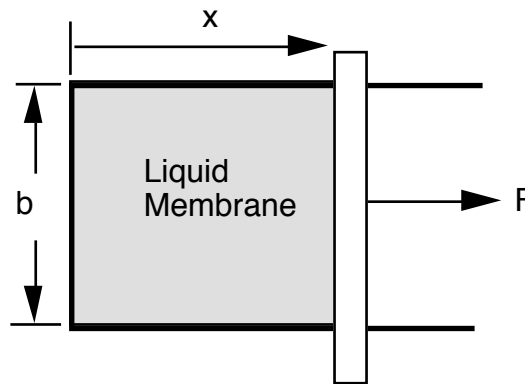
The free energy is a function of the length of the membrane, $U(x)$. When the slider moves by a distance δx, the area of the membrane increases by $2b\delta x$, and the surface energy increases by $2\gamma b\delta x$. When the slider moves by a distance δx, the weight drops by the same distance, so that the gravitational energy *decreases* by $F\delta x$. The net energy change is the combination of the two effects:

$$\delta U = \left(2\gamma b - F\right)\delta x .$$

The two terms compete. The surface energy decreases when the slider moves to the left. On the other hand, the gravitational energy decreases when the slider moves to the right.

The thermodynamic system reaches a state of equilibrium when the variation in the free energy associated with the variation in the position of the slider vanishes, namely, when

$$F = 2b\gamma .$$

By measuring the weight F that equilibrates the membrane, one determines the surface energy density. The surface energy density behaves like a force per unit

length.   Perhaps for this reason the quantity γ is also known as the surface tension.

A drop of water on a ceiling: a competition between surface energy and gravity. Let a be the radius of a drop of water on a ceiling. The gravity tends to pull the drop down, but the surface tension tends to keep it up. In this case, the surface tension of the surface molecules acts like a rubber bag, holding the water inside. Determine the maximum drop size.

Balance the forces due to surface tension and gravity, we obtain

$$\gamma a \approx \rho g a^3 \, .$$

The competition between the gravity and the surface energy defines a length scale:

$$a = \sqrt{\frac{\gamma}{\rho g}}$$

Estimation of the maximum size of a drop on the ceiling. Take γ = 0.1 J/m², ρ = 1000 kg/m³, and g = 10 m/s². We find that a is of magnitude of cm.

Measuring the surface energy of a solid.   Experimental determination of the surface energy of a solid is challenging. To measure the surface energy, one has to observe a phenomenon in which fresh surface is created. For a liquid, the fresh surface is created by flow, drawing molecules from the interior of the liquid to the surface. For a solid, one has to do essentially the same thing. That is, one has to arrange a situation in which additional atoms move from the interior to the surface. Heat the solid so that it creeps (i.e., flows slowly). In the following words, Griffith described how he measured the surface energy of glass.

Between 730° C and 900° C the method described below was found to be practicable. Fibers of glass about 2 inches long and from 0.002-inch to 0.01-inch diameter, with enlarged spherical ends, were prepared. These were supported horizontally in stout wire hooks and suitable weights were hung on their mid-points. The enlarged ends prevented any sagging except that due to extension of the fibers. The whole was placed in an electric resistance furnace maintained at the desired temperature. Under these conditions viscous stretching of the fiber occurred until the suspended weight was just balanced by the vertical components of the tension in the fiber. The latter was entirely due, in the steady state, to the surface tension of the glass, whose value could therefore be calculated from the observed sag of the fiber. In the experiments the angle of sag was observed through a window in the furnace by means of a telescope with a rotating cross wire. If w is the suspended weight, d the diameter of the fiber, T the surface tension, and θ the angle at the point of suspension between the two halves of the fiber, then, evidently,

$$\pi dT \sin \frac{\theta}{2} = w .$$

Measuring the surface energy of a solid by zero creep experiment. Udin et al. (1949) described an experimental setup based a similar principle. A series of small weights of increasing magnitudes are suspended from an array of copper wires of uniform cross section. This array is brought to a temperature at which creep is appreciable. If the weight overbalances the contracting effect of surface tension, the wire elongates; otherwise, it shrinks. The rate of creep is zero when the weight balances the contracting effect of surface. This condition of thermodynamic equilibrium is

$$w = \pi r \gamma ,$$

where w is the hanging weight, and r is the radius of the copper wire. In the experiment, wires under larger weights elongate, and wires under smaller weights shrink. Some intermediate weight just balances the contracting effect of the surface tension. A more recent paper by Josell and Spaepen (1993) described more applications of this zero creep experiment.

- H. Udin, A.J. Shaler, J. Wulff, The surface tension of solid copper. Transactions of the American Institute of Mining and Metallurgical Engineers 185, 186-190 (1949).
- H. Udin, Surface tension of solid copper, PhD Thesis, Massachusetts Institute of Technology (1949). This thesis is available online at http://dspace.mit.edu/bitstream/handle/1721.1/61810/29584462.pdf?sequence=1
- D. Josell and F. Spaepen, Determination of the interfacial tension by zero creep experiments on multilayers. Acta Metallurgica Materialia 41, 3007-3027 (1993).

Deriving the condition of equilibrium. The condition of equilibrium $w = \pi r \gamma$ itself is interesting. If you think of the surface tension as the force per unit length, then the balance of the force with the weight gives you $w = 2\pi r \gamma$. What is wrong with the factor of 2?

We can obtain the correct result by starting from the first principles. The wire and the weight together form a thermodynamic system. The surface of the wire contributes free energy $\gamma \left(2\pi r l + 2\pi r^2 \right)$, where r is the radius of the wire and l the length of the wire. Here the surface of the wire includes the cylindrical surface and the two ends of the wire. The weight contributes potential energy $-wl$. The free energy of the system—the wire and the weight together—is

$$U = \gamma \left(2\pi r l + 2\pi r^2 \right) - wl .$$

When the wire elongates, atoms from the interior of the wire move to the surface, and the volume of the wire remains the same. The volume of the wire is $\pi r^2 l = V$. Rewrite the free energy of the wire as

$$U = \gamma \left(2\sqrt{\pi V l} + 2\frac{V}{l} \right) - wl \,.$$

The system has a single degree of freedom: the length of the wire. The free energy is a function of a single variable, $U(l)$. The system reaches thermodynamic equilibrium when $dU/dl = 0$, namely,

$$\gamma \left(\sqrt{\frac{\pi V}{l}} - 2\frac{V}{l^2} \right) = w \,.$$

Replace V in this result, and we obtain that

$$\gamma \left(\pi r - 2\pi r \frac{r}{l} \right) = w$$

Because $r << l$, we drop the second term in the bracket, reaching
$$\pi r \gamma = w \,.$$

      Why can't we directly use the balance of forces? The surface of the wire is curved. Consequently, the surface tension causes the Laplace pressure to atoms inside the wire:

$$p = \frac{\gamma}{r} \,.$$

The weight needs to balance the force in the skin and the pressure in the interior:
$$w = 2\pi r \gamma - \pi r^2 p \,.$$

This result recovers $w = \pi r \gamma$.

The Griffith Theory

      A large sheet of a glass is under stress σ. The sheet has unit thickness. In the reference state, the stressed sheet contains no crack, and is in a state of homogeneous deformation. In the current state, the stressed sheet contains a crack of length $2a$, and is in a state of inhomogeneous deformation.

      Helmholtz free energy. The sheet containing the crack is a thermodynamic system. The system is taken to be in thermal equilibrium with a reservoir of energy of a fixed temperature. For the time being we assume that the displacement at the loading point is fixed, so that the loading device does no work on the system when the length of crack changes.

      Let U be the Helmholtz free energy of the system relative to that in the reference state. The length of the crack, a, is the thermodynamic variable of the system, and the free energy of the system is a function of the thermodynamic variable, $U(a)$. Thermodynamics dictates that the system should evolve in the

direction that reduces the Helmholtz free energy. That is, the length of the crack a should change in the direction that reduces the Helmholtz free energy $U(a)$.

Calculate the Helmholtz free energy of the system as a function of the length of the crack. We next calculate the difference in energy between the two states. Let γ be the surface energy per unit area. Between the reference and the current states, the difference in surface energy is $4a\gamma$.

For the time being, assume that the loading grips are rigidly held, so that the displacement is fixed, and the loading device does no additional work after a fixed displacement is applied. When the crack is introduced into the body, the elastic energy reduces. To determine the amount of the reduction, one has to solve the boundary-value problem to determine the inhomogeneous deformation in the current state. This difficult elasticity problem is for professional elastitians. Look how complicated the stress field must be near the crack. Griffith used the elasticity solution of Inglis, because a crack is just a special case of an ellipse when $b/a \rightarrow 0$. This part of the Griffith paper is difficult to read, and is uninteresting. In the end he made small errors. It would be a distraction for us to go through the process of solving this boundary-value problem here.

An alternative approach is to invoke linearity and dimensional considerations. For a linearly elastic boundary-value problem, the stress field is linearly proportional to the applied stress. The elastic energy per unit volume is proportional to σ^2/E. The elastic energy in an infinite sheet is infinite. However, we are interested in the difference in elastic energy between the cracked sheet and the uncracked sheet. Note that the crack length a is the only length scale in the boundary-value problem. Consequently, the difference in elastic energy between the two sheets takes the form

$$\beta \frac{\sigma^2}{E} a^2,$$

where β is a numerical value. Thus, from very basic considerations, we get nearly everything except for a pure number. This number must be determined by solving the boundary-value problem in linear elasticity. The solution turns out to be $\beta = \pi$. You can find the solution of the full problem in Timoshenko and Goodier.

Relative to the uncracked sheet, in the cracked sheet the combined surface energy and elastic energy is

$$U(a) = +4\gamma a - \pi \frac{\sigma^2}{E} a^2.$$

The density of surface energy γ and the applied stress σ are taken to be constant for the time being. The crack length, $2a$, is the thermodynamic variable. As expected, when the crack length increases, the surface energy increases, but the elastic energy decreases.

Plot the free energy as a function of the crack length. The free energy first goes up, reaches a peak, and then goes down. Because there is no minimum free energy, the crack cannot reach equilibrium.
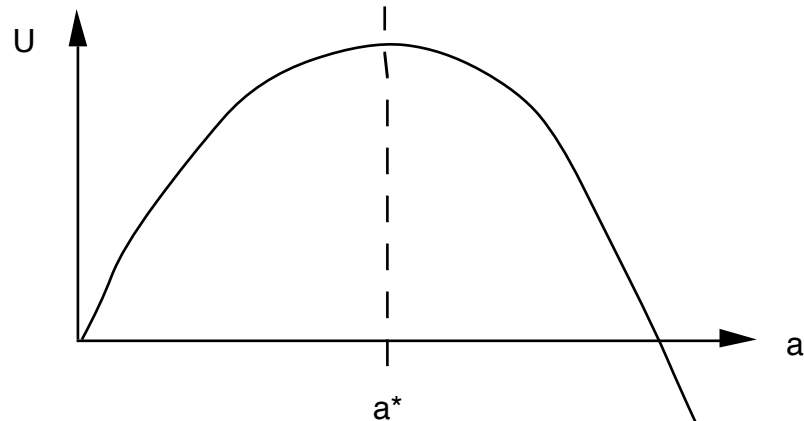
When the length of the crack changes by δa, the free energy changes by

$$\delta U = 2\left(2\gamma - \pi \frac{\sigma^2 a}{E}\right)\delta a$$

Let us distinguish two situations.

Small crack heals. If the crack is small, namely,

$$2\gamma > \pi \frac{\sigma^2 a}{E} \, ,$$

the surface energy prevails over the elastic energy. To reduce the free energy, $\delta U < 0$, the crack must decrease its length, $\delta a < 0$. The crack does so by healing, i.e., forming atomic bonds one by one, like closing a zipper.

In reality crack healing is not often observed. This is not because the thermodynamics is wrong, but because surfaces are not flat to the atomic dimension, so that atoms cannot meet across the gap and form bonds. Several examples show that a crack heals under certain conditions.

- *Adhesives.* Soft material can heal readily by flows.
- *Wafer bonding.* If the surfaces are indeed made flat, they will join.
- *Sintering.* At elevated temperatures, atoms can diffuse, so that the two surfaces change shape and can join.

Large crack grows. If the crack is large, namely,

$$2\gamma < \pi \frac{\sigma^2 a}{E} \, ,$$

the elastic energy prevails over the surface energy. To reduce the free energy, $\delta U < 0$, the crack must increase its length, $\delta a > 0$. The crack does so by breaking atomic bonds one by one, like opening a zipper. This is the situation studied in this course.

The Griffith Experiments

The main prediction of the Griffith theory can be written as

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi a}} \ .$$

He performed several experiments to ascertain various predictions of the equation.

Experiment 1. Confirm that

$$\sigma_c \sqrt{a} = \text{constant} \ ,$$

independent of the size of the crack. Griffith invented the following experiment, which would be repeated to this day. Start with several sheets glass (large spherical bulbs actually). Introduce a crack in each sheet either with a glass-cutter's diamond, or by scratching with a hard steel edge and tapping gently. Measure the strength of each sheet. Two important points:

(1) The crack introduced is in the mm to cm range, much longer than any "natural flaws" in the sheets, so that the natural flaws are negligible. In this way Griffith circumvented the uncertainties associated with the natural flaws.

(2) The cracks introduced in different sheets have different lengths, and the measured strengths are also different.

His data confirmed that $\sigma_c \sqrt{a} = \text{constant}$.

	Crack Length, $2a$ mm	Measured Strength, σ_c MPa	$\sigma_c \sqrt{a}$ MPa$\sqrt{\text{m}}$
sample 1	3.8	6.0	0.26
sample 2	6.9	4.3	0.25
sample 3	13.7	3.3	0.27
sample 4	22.6	2.5	0.27

(Data from the Griffith experiment)

Experiment 2. Confirm that the constant is indeed $\sqrt{2\gamma E / \pi}$. Young's modulus for the glass used by Griffith was E = 62 GPa. The surface energy inferred from the measured breaking strength is $\gamma = 1.75 \text{J/m}^2$.

Griffith needed an independent measurement of the surface energy. He did the creeping fiber experiment. The value he obtained was $\gamma = 0.54 \text{J/m}^2$. The agreement was fair.

Griffith's error. Griffith made a mistake in his calculation of the elastic energy, and gave a wrong formula:

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi v a}} \quad \text{(wrong formula)}$$

where v is Poisson's ratio, and v = 0.251 for glass. This erroneous result estimates a value of the surface tension $\gamma = 0.44 \text{J/m}^2$. This value determined from his fracture experiment was lower than the surface determined from his zero creep experiment, $\gamma = 0.54 \text{J/m}^2$. Griffith tried to find an explanation for this erroneous estimate. I do not believe that one can find a sound explanation when the surface tension determined by fracture experiment is smaller than that determined by the zero-creep experiment. But this is a moot point: Griffith made an error in his formula.

Experiment 3. Measure strengths of glass fibers.
Experimental strength. For a fixed pre-existing crack size a, there is a critical stress:

$$\sigma_c = \sqrt{\frac{2\gamma E}{\pi a}} \;.$$

This is the stress needed to fracture the sample. This relation shows that the fracture strength depends on the crack size. Because different samples have different crack sizes, the fracture strength is not a material property. The measured strength has large scatter. Take representative values γ = 1 J/m², E = 10¹¹ N/m², a = 10⁻⁶ m, the strength is 250 MPa. This corresponds to the experimental strength.
Theoretical strength. Assume that the solid is flawless. There is no stress concentration. The solid breaks when the applied stress is so high to break atomic bonds. If we put $a = 10^{-10} \text{m}$ (atomic dimension) into the above formula, we obtain an estimate of the theoretical strength ~10 GPa.

Alternatively, a commonly quoted rough estimate of the theoretical strength is

$$S_{th} = \frac{E}{10} \;.$$

Today the theoretical strength can be calculated by atomistic simulations.

Griffith measured the strength of glass fibers of diameters between 107 μ m and 3.3 μ m. The data scattered, but the trend was that the strength increased as the fiber diameter decreased. He reported a value of strength of 171 MPa for a fiber of diameter 107 μ m, and a value of strength of 3.4 GPa for a fiber of diameter 3.3 μ m. The theoretical strength of the glass corresponds to a fiber

of the smallest possible (molecular) diameter. He extrapolated his data to the molecular diameter, and estimated the theoretical strength to be $12\,\text{GPa}$.

Historical Notes

Alan Arnold Griffith (1893-1963). He was born in London on 13 June 1893. He earned his B.Eng. in mechanical engineering in 1914, M.Eng. in 1917, and D.Eng. in 1921, all from the University of Liverpool. In 1915, he entered the Royal Aircraft Factory (later known as the Royal Aircraft Establishment), and advanced through a workshop traineeship followed by other positions to become senior scientific officer in 1920. In 1917, together with G.I. Taylor, he published a pioneering paper on the use of soap films in solving torsion problems, and in 1920 he published his famous paper on the theory of brittle fracture. He then worked on the design theory of gas turbines. Griffith was Head of the Engine Department of the Royal Aircraft Establishment in 1938 and joined Rolls Royce as research engineer in 1939. He worked first on conceptual design of turbojet engines and later on vertical takeoff aircraft design. He retired in 1960 but continued working as a consultant for Rolls Royce. He died on 13 October 1963.

- Wikipedia page http://en.wikipedia.org/wiki/Alan_Arnold_Griffith
- A.A. Rubbra, "Alan Arnold Griffith 1893-1963". *Biographical Memoirs of Fellows of the Royal Society* **10**: 117–126. doi:10.1098/rsbm.1964.0008

Inglis (1913) vs. Griffith (1921). Rewrite Inglis's equation as

$$\sigma_c \sqrt{a} \approx \frac{1}{2}\sigma_{\max}\sqrt{\rho}\ .$$

We have interpreted $\sigma_{\max}$ as the theoretical strength of atomic bonds, and ρ as the atomic spacing. The Griffith theory gives that

$$\sigma_c \sqrt{a} = \sqrt{\frac{2\gamma E}{\pi}}\ .$$

The two theories give the same prediction: $\sigma_c \sqrt{a}$ is a constant independent of the size of the crack. Griffith did experiments to confirm that $\sigma_c \sqrt{a}$ is indeed the constant.

In the Inglis theory, the constant involves atomic strength and atomic size. In the Griffith theory, the constant involves Young's modulus and surface energy. If we adopt any simple-minded atomic model, we can show that the two constants are essentially the same. See below.

Both theories work approximately for silica glass. Neither works for steel. Both theories survived to this day, in somewhat different forms. In general terms, the Inglis theory has evolved into the stress approach to fracture, and the Griffith theory has evolved into the energy approach. We will talk more about both approaches in coming lectures.

Polanyi (1921) derived an approximate relation between theoretical strength, modulus, surface tension and atomic spacing. If a solid is extended uniformly, the extension should be stable until the stress reaches the theoretical strength σ_{max}. Assume that Hooke's law holds up to fracture. The elastic energy per atom at fracture is $b^3 \left(\sigma_{max} \right)^2 / 2E$, where b is the atomic spacing. After fracture, the surface energy per atom is γb^2. The elastic energy in the stressed specimen must provide the surface energy. Equating the two quantities gives that

$$\frac{\left(\sigma_{max} \right)^2 b}{2E} \approx \gamma$$

Taking representative values $b = 10^{-10}\,\text{m}$, $E = 10^{11}\,\text{N/m}^2$ and $\gamma = 1\,\text{N/m}$, we estimate that $\sigma_{max} \approx 40\,\text{GPa}$.

- Von M. Polanyi, On the nature of the tearing process. Zeitschrift für Physik 7, 323-327 (1921)
- E.P. Wigner and R.A. Hodgkin. Michael Polanyi (1891-1976). Biographical Memoirs of Fellows of the Royal Society 23, 413–448 (1977). doi: 10.1098/rsbm.1977.0016.