Instabilities

References

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These notes are written to remind myself of what to say in class. They are rough in many places, and I'll refine them when I have time. No errors are deliberately planted, but errors are inevitable, both in judgment and in fact. Please kindly let me (suo@seas.harvard.edu) know if your find any.

Free energy and generalized coordinate. Equilibrium and stability. As an example, consider a system consisting of a rod, a weight and a spring. We describe the configuration of the system with a single degree of freedom: the angle θ. The free energy G of the system consists of the potential energy of the weight and that of the spring. The rod is rigid and weightless. The weight lowers its energy by rotating the rod. The spring lowers its energy by keeping the rod vertical. The two factors compete to determine the outcome: the rod stands vertical when the weight is small, and rotates when the weight is large.
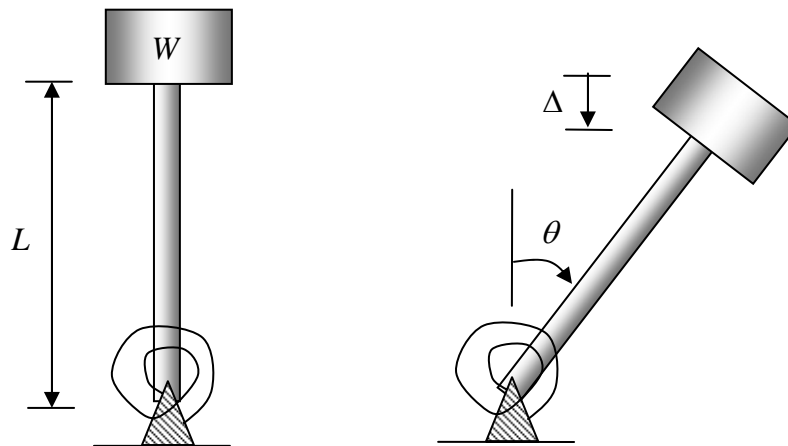
In general, for a system of one degree of freedom, we specify the configuration of the system with a generalized coordinate q, which is a parameter uncontrolled and free to vary. The free energy of the system is a function of the generalized coordinate, $G(q)$. A stable equilibrium configuration corresponds to a generalized coordinate that minimizes the free energy.

The curve in the figure shows several stationary configurations: two minima, a maximum, and an inflexion point. A configuration locally minimizes the free energy is also known as a metastable equilibrium configuration. A stable equilibrium configuration corresponds to a *global minimum*.

G

maximum

Fixed λ

inflexion

local
minimum

global
minimum

q
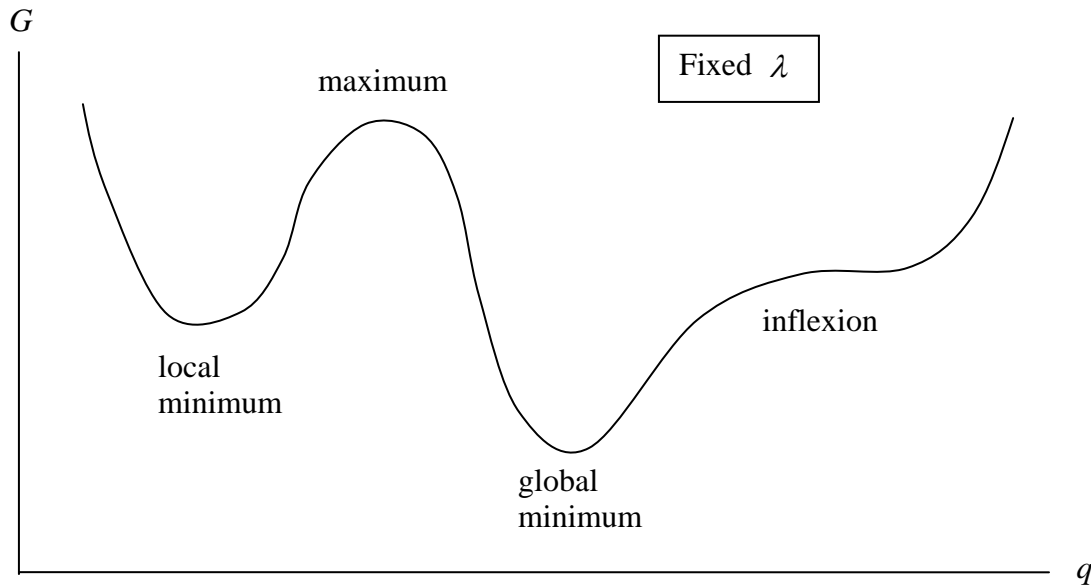
Control parameter. We can modify the system, say, by altering the weight, the stiffness of the spring, or the length of the rod. These modifications lead to *a family of systems*. Each member of the family has its own free-energy function. To formalize this idea, we introduce a parameter λ, which can be the weight, or the stiffness of the spring, or the length of the rod. In general, we call λ the control parameter. The free energy of this family of systems is a function of the generalized coordinate and the control parameter, $G(q, \lambda)$.

A member system in this family has a fixed λ. The same principle of equilibrium and stability applies to this system: When λ is fixed, and q is allowed to vary, of all configurations of the system, the stable equilibrium configuration is the one that the generalized coordinate q minimizes G, while λ is held fixed.

Configurational transitions of two types. Now we wish to vary the control parameter to study the family of systems as a whole. For a fixed λ, the member system reaches an equilibrium configuration characterized by a value of the generalized coordinate, q_E. As we go from one system to another by gradually changing λ, the equilibrium configuration q_E often changes gradually, but not always. A critical value λ_c may exist, such that when the control parameter λ passes λ_c, the systems on the two sides of λ_c have drastically different equilibrium configurations. This phenomenon is called phase transition or configurational transition. Familiar examples include meting of ice when the temperature passes the melting point, and buckling of a column when the load passes a critical value. Configurational transitions may be classified into two types.

Configurational transition of the first type. Consider a family of systems of the following characteristics. Each member system with a fixed λ has two local minima. The relative position of the two minima is significant: the lower minimum corresponds to the stable equilibrium, and the higher one corresponds to the metastable equilibrium. As λ changes gradually, the locations of the two minima move gradually. When λ reaches a critical value λ_c, the two minima reach the same height. A further change in λ will switch the relative position of the two minima. The figure illustrates an example. At $\lambda_1 < \lambda_c$, the minimum on the left corresponds to the stable equilibrium configuration. At $\lambda_2 > \lambda_c$, the minimum on the right corresponds to the stable equilibrium configuration.
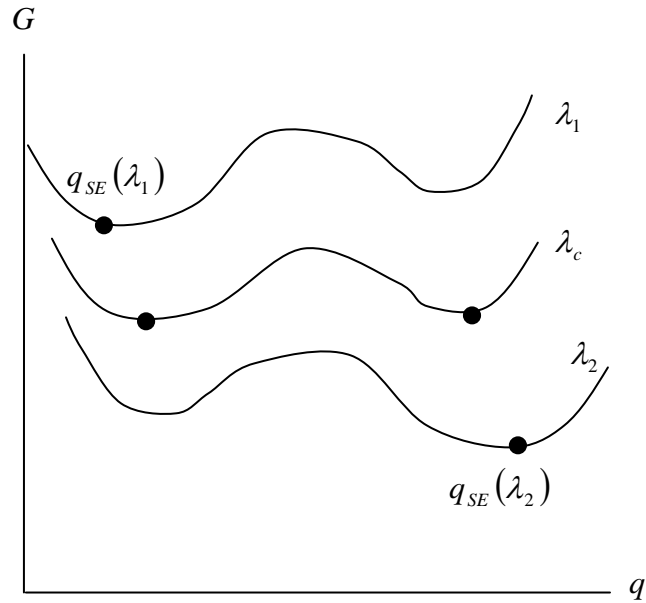
To switch from the configuration of a local minimum to the configuration of the global minimum, the system has to overcome an energy barrier. If a perturbation in energy is available, from thermal fluctuation or from any other external intervention (shaking, plucking, etc.), then such transition will occur. If the perturbation in energy is unavailable, the system will be trapped in the configuration of local minimum.
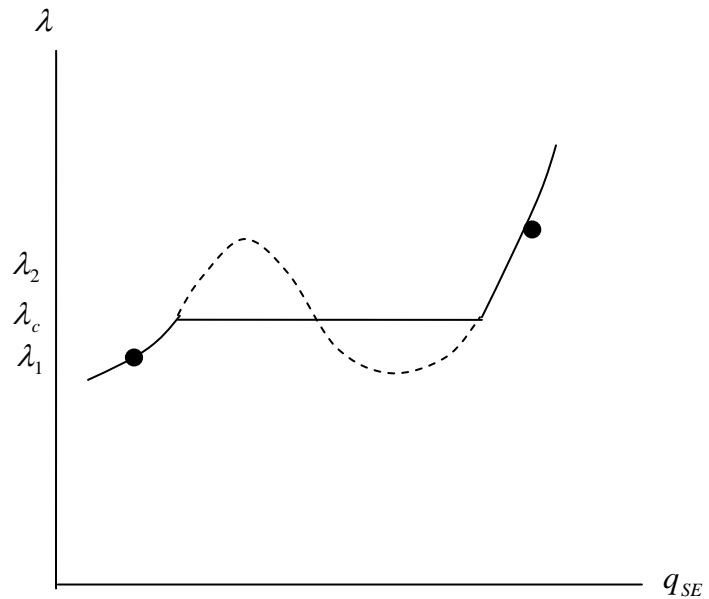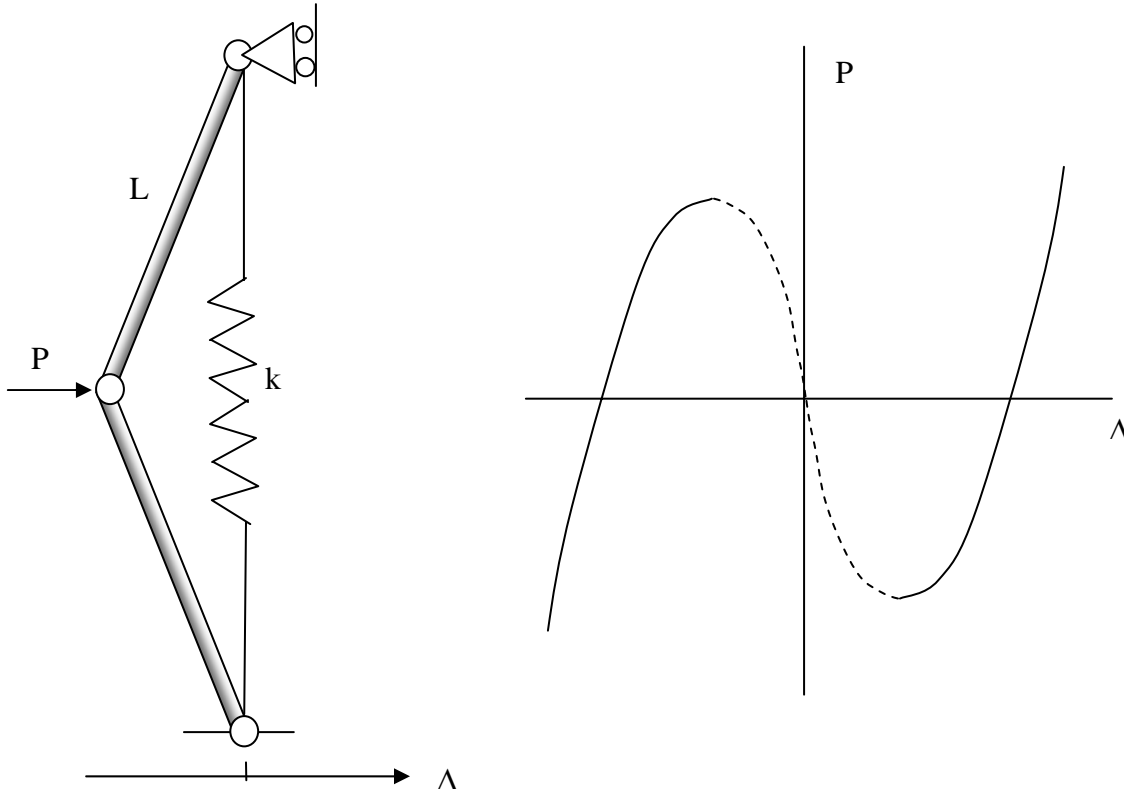
Configurational transition of the second type. Next consider a family of systems with the following characteristics. When $\lambda < \lambda_c$, the system has a minimum. As λ increases gradually, the depth of the minimum becomes smaller and smaller. Then a critical value λ_c is reached, such that the local minimum disappears. In this situation, there will be no energy barrier for the system to switch configuration.

Example. Here is an example that exhibits both types of transitions (Thompson and Hunt, 1984). You can regard the displacement Δ as the generalized coordinate, and the force P as the control parameter. You can readily write an expression of the energy function of the system, but the qualitative behavior should be intuitive. When no force is applied, the energy of the system has two minima of equal height, so that zero force is the critical load for a configurational transition of the first type. The system may switch between the two

configurations if you pluck it.

When a force is applied to push to the right, a term $-\Delta P$ is added to the energy. Consequently, the minimum on the left goes up, and the minimum on the right goes down. When the force reaches some critical value, the minimum on the left disappears, and the structure will snap to the right. This is a configurational transition of the second type.

Critical point of configurational transition of the second type. **Bifurcation analysis**. Let us focus on configurational transitions of the second type, and summarize the procedure to determine the equilibrium configuration and critical point. The analysis is local, so that each minimum is understood to be a local equilibrium and corresponds to a metastable configuration. We will not always explicitly mention the words local and metastable.

We will be dealing with systems that have a smooth free-energy function, $G(q, \lambda)$, so that we can use the Taylor series to as high a power as we please. First let us look at a single member in the family of systems; that is, we fix the control parameter λ. When the configuration of the system changes slightly from q to $q + \delta q$, the free energy changes by

$$G(q + \delta q, \lambda) - G(q, \lambda) = \frac{\partial G(q, \lambda)}{\partial q} \delta q + \frac{1}{2} \frac{\partial^2 G(q, \lambda)}{\partial q^2} (\delta q)^2 .$$

For a configuration to be stationary, the first derivative vanishes:

$$\frac{\partial G(q, \lambda)}{\partial q} = 0 .$$

For a certain family, it is possible that some of these equilibrium configurations correspond to local minima of G (i.e., $\partial^2 G(q, \lambda)/\partial q^2 > 0$) but others correspond to local

maxima (i.e., $\partial^2 G(q,\lambda)/\partial q^2 < 0$). When this happens, because G is a smooth function, there must be a system with a particular value of λ such that the second derivative vanishes:

$$\frac{\partial^2 G(q,\lambda)}{\partial q^2} = 0 .$$

The above two equations are solved simultaneously to determine the critical point (λ_c, q_c).

An equilibrium configuration q_E is a property of a member system with a fixed λ. A critical point (λ_c, q_c) is a property of the whole family of systems. Of course, some member system may have no equilibrium configuration or may have multiple equilibrium configurations. Some family of systems may have no critical point or have multiple critical points.

Behavior of a family of systems near a critical point. **Post-bifurcation analysis**. Let (λ_c, q_c) be a critical point of a family of systems, and let (λ, q) be a point in the neighborhood of the critical point. We expand the free-energy function into the Taylor series around the critical configuration q_c:

$$G(q,\lambda) = A(q - q_c) + B(q - q_c)^2 + C(q - q_c)^3 + D(q - q_c)^4 .$$

We have omitted the zeroth order term because it does not affect equilibrium configurations. We can go to even higher orders if we need to. The coefficients A, B, C and D are in general nonlinear functions of λ, but are independent of q.

The critical point (λ_c, q_c) satisfies $\partial G / \partial q = 0$ and $\partial^2 G / \partial q^2 = 0$, so that

$$A(\lambda_c) = 0, \quad B(\lambda_c) = 0 .$$

Unless the family of systems have some special symmetry, typically we should have $C(\lambda_c) \neq 0$ and $D(\lambda_c) \neq 0$. Consequently, to the leading order in $\lambda - \lambda_c$, A and B are linear in $\lambda - \lambda_c$, but C and D are constants. We write

$$G(q,\lambda) = a(\lambda - \lambda_c)(q - q_c) + b(\lambda - \lambda_c)(q - q_c)^2 + c(q - q_c)^3 + d(q - q_c)^4 ,$$

where a, b, c and d are constants independent of q and λ. This expression is still very general and its physical consequence is difficult to survey. In what follows, we discuss several representative cases. In effect, we sub-classify the behavior near the critical point of configurational transitions of the second type.

Snapping (The case $a \neq 0, c \neq 0$). In the above expression of the energy function, the second term is of a higher order than the first term, and can be neglected. The fourth term is of a higher order than the third term, and can be neglected. In this case, the free-energy function to the leading order of $(q - q_c)$ and $\lambda - \lambda_c$ is

$$G(q,\lambda) = a(\lambda - \lambda_c)(q - q_c) + c(q - q_c)^3 .$$

We next examine equilibrium configurations in the neighborhood of the critical point (λ_c, q_c). Without losing generality, we adopt the sign convention such that a stable equilibrium configuration satisfies that $\lambda < \lambda_c$ and $q < q_c$. This sign convention is equivalent to assuming that $a < 0$ and $c < 0$. When $(q - q_c)$ is small, the free energy is dominated by the term linear in $(q - q_c)$. When $(q - q_c)$ is large, the free energy is dominated by the term cubic in $(q - q_c)$. For a member system with $\lambda < \lambda_c$, the slope of G at q_c is positive, so that a minimum and a maximum exit. For a member system with $\lambda > \lambda_c$, the slope of G at q_c is negative, so that a G

is a monotonic decreasing function of q.
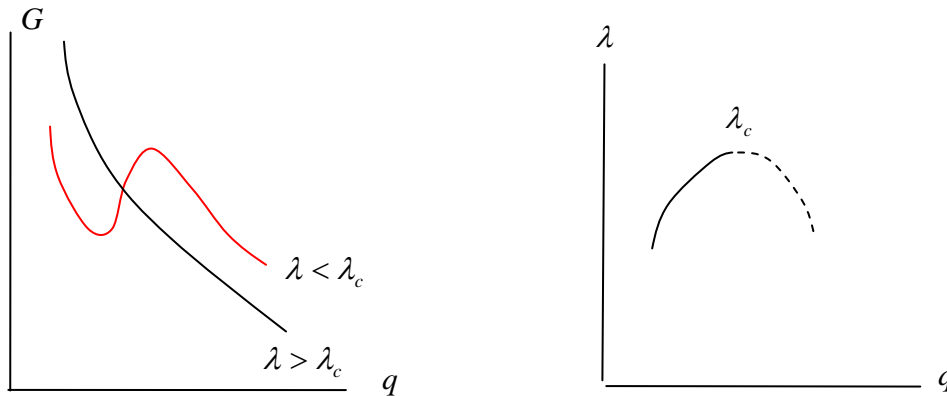
Each minimum corresponds to a stable equilibrium configuration, and each maximum corresponds to an unstable equilibrium configuration. Setting $\partial G / \partial q = 0$, we obtain that

$$\lambda = \lambda_c - \frac{3c}{a}(q - q_c)^2 .$$

This equation relates the control parameter and the generalized coordinate of the equilibrium configurations. This $\lambda - q$ curve exhibits a maximum at (λ_c, q_c). A point on the left side of the curve corresponds to a local minimum, and a point on the other side of the curve corresponds to a local maximum. If the actual λ exceeds the maximum λ_c, there will be no equilibrium configuration near q_c, and the structure is expected to jump to a distant configuration. This kind of instability is called *snapping instability or limit-point instability*.

Symmetry breaking (The case $a = 0, c = 0$). The example in the beginning of the notes has a special feature. We can construct the structure with a perfect symmetry such that the free energy is an even function of the generalized coordinate q. Consequently, $q = 0$ is an equilibrium configuration for all values of the control parameter λ. Let λ_c be the critical value. In the neighborhood of the critical point, the free-energy function takes the form

$$G(q, \lambda) = b(\lambda - \lambda_c)q^2 + dq^4 .$$

Setting $\partial G / \partial q = 0$ and obtain that
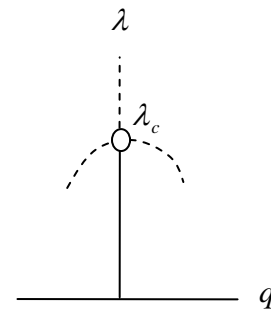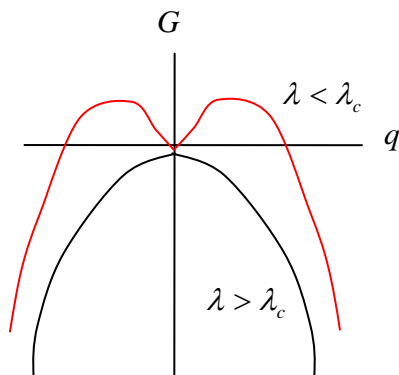
$$2b(\lambda - \lambda_c)q + 4dq^3 = 0 .$$

This gives the trivial equilibrium configuration $q = 0$, as well as two nontrivial equilibrium configurations:

$$\lambda = \lambda_c - \frac{2d}{b}q^2 .$$

When q is small, the free energy is dominated by the term quadratic in q. When q is large, the free energy is dominated by the q^4 term. We will still adopt the sign convention that stable configurations exist for $\lambda < \lambda_c$. This convention is equivalent to setting $b < 0$. Because of the symmetry, there is no need to adopt any sign convention for q. We need to look at two subcases: $d > 0$ and $d < 0$.
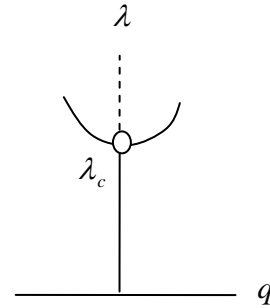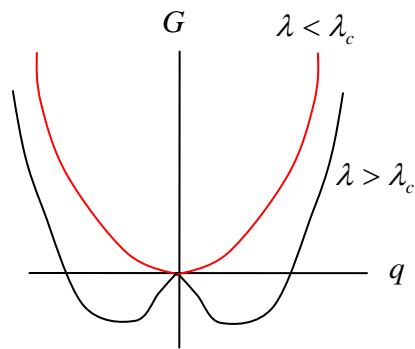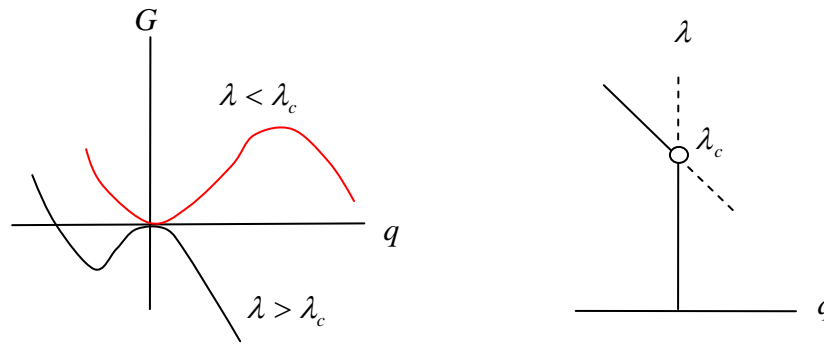
Subcase $d > 0$. For a member system with $\lambda < \lambda_c$, the configuration $q = 0$ is a local minimum. For a member system with $\lambda > \lambda_c$, the configuration $q = 0$ is a local maximum, but two new local minima emerged. This kind of instability is called the *stable bifurcation*. Example: ferroelectric crystal.

Subcase $d < 0$. For a member system with $\lambda < \lambda_c$, the configuration $q = 0$ is a local minimum, and the nontrivial equilibrium configurations correspond to two maxima. For a member system with $\lambda > \lambda_c$, the configuration $q = 0$ becomes a maximum. This kind of instability is called *unstable bifurcation*.

The case $a = 0, c \neq 0$. The system is asymmetric, but $q = 0$ is an equilibrium state for all values of λ. This case often arises in elastic structures. In the example in the beginning of the notes, the spring may be asymmetric with respect to the two directions of rotation. To the leader order in q and $\lambda - \lambda_c$, the free-energy function is

$$G(q, \lambda) = b(\lambda - \lambda_c)q^2 + cq^3.$$

Setting $\partial G / \partial q = 0$, we obtain that

$$2b(\lambda - \lambda_c)q + 3cq^2 = 0.$$

This gives the trivial equilibrium configuration $q = 0$, as well as a nontrivial equilibrium configuration,

$$\lambda = \lambda_c - \frac{3c}{2b}q.$$

We adopt the sign convention $b < 0$ and $c < 0$. When q is small, the free energy is dominated by the term quadratic in q. When q is large, the free energy is dominated by the term cubic in q. For a member system with $\lambda < \lambda_c$, the configuration $q = 0$ is a local minimum, and there is a local maximum on the right side.. For a member system with $\lambda > \lambda_c$, the configuration $q = 0$ becomes a local maximum, and there is a local minimum on the right side.

The case $a \neq 0, c = 0$. For completeness, we note the free-energy function

$$G(q,\lambda) = a(\lambda - \lambda_c)q + dq^4.$$

The analysis is similar to that of the other cases. (I don't know if this case describes any significant physical situation.)

Load-displacement relation near a critical point. When the control parameter λ is a generalized force, the free energy takes the form:

$$G(q,\lambda) = U(q) - \lambda \Delta(q),$$

where $U(q)$ is the internal energy of the structure, and $\Delta(q)$ is the generalized displacement that is work-conjugate to λ. To obtain the generalized displacement near a critical point, we can compare this expression of potential energy with any of the previous expression. For example, in the case $a = 0, c = 0, b > 0, d > 0$, we obtain that

$$\Delta = -bq^2.$$
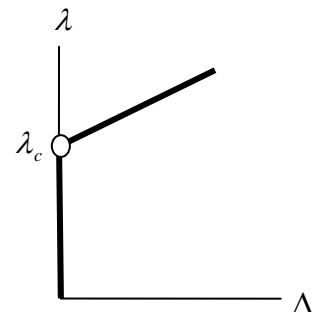
Recall that

$$\lambda = \lambda_c - \frac{2d}{b}q^2$$

Eliminating q from the two equations, we obtain that

$$\lambda - \lambda_c = \frac{2d}{b^2}\Delta$$

When $\lambda > \lambda_c$, this is the load-displacement relation near the critical

point, and $2d/b^2$ is the stiffness of the structure. When $\lambda < \lambda_c$, however, $q = 0$, $\Delta = 0$ and the stiffness of the structure is infinite. This infinite stiffness results from our model, in which only one degree of freedom is allowed.
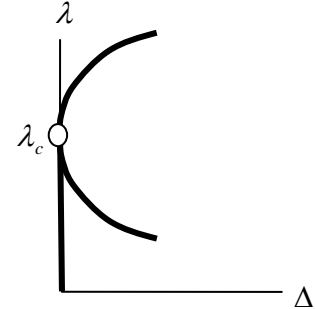
As another example, consider the case $a = 0, b < 0, c < 0, d = 0$. In this case

$$\Delta = -bq^2$$

$$\lambda = \lambda_c - \frac{3c}{2b}q$$

Eliminating q, we obtain the load-displacement relation near the critical point:

$$\lambda - \lambda_c = \pm\frac{3c}{2b}\sqrt{\frac{\Delta}{-b}} .$$

Koiter's theory of imperfection sensitivity. Consider a family of systems that have a perfect symmetry such that $q = 0$ is an equilibrium configuration for all values of the control parameter λ. Of course, this perfection is rarely achieved in reality. For an imperfect structure, $q = 0$ is no longer an equilibrium configuration for all values of the control parameter. When the magnitude of the imperfection is small, the imperfect structure can be viewed as a perturbation from the perfect structure. When the perfect structure is asymmetric, the load-displacement curves of the imperfect structure may look like one of these. In one case, the load-displacement curve keeps increasing, so that the structure is said to be insensitive to this type of imperfection.

In the other case, the load-displacement curve exhibits a maximum. If the actual load exceeds this maximum, there will be no equilibrium state with small q, and the system is expected to jump to a distant equilibrium configuration. Consequently, the imperfect system may exhibit this *snapping instability* or *limit-point instability* Denote the critical point of the imperfect system by (λ_s , q_s). The asymmetric structure is said to be sensitive to this type of imperfection.
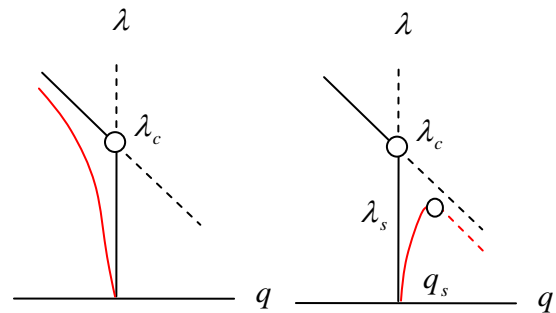
To quantify the effect of imperfection, let $\varepsilon > 0$ be the amplitude of imperfection. The free energy becomes a function of three variables, $G(q, \lambda, \varepsilon)$. When $\varepsilon = 0$, the system is perfect and $q = 0$ is an equilibrium configuration for all values of λ. When $\varepsilon > 0$, $q = 0$ is no longer an equilibrium configuration. Consequently, to the leading orders in q, $(\lambda - \lambda_c)$ and ε, the imperfect system has the free-energy function

$$G(q, \lambda, \varepsilon) = b(\lambda - \lambda_c)q^2 + cq^3 + \alpha\varepsilon q,$$

where *b, c,* and α are constants independent of q, λ and ε. We follow the sign convention $b < 0$ and $c < 0$. As we will see, when $\alpha < 0$, the family of systems exhibits a limit point.

To determine the critical point of this imperfect family of systems, we set $\partial G/\partial q = 0$ and $\partial^2 G/\partial q^2 = 0$, so that

$$2b(\lambda - \lambda_c)q + 3cq^2 + \alpha\varepsilon = 0.$$

and

$$2b(\lambda - \lambda_c) + 6cq = 0.$$

The combination of the above gives the new location of the critical point of the family of imperfect systems:

$$q_s = \left(\frac{\alpha\varepsilon}{3c}\right)^{1/2}, \quad \lambda_s = \lambda_c + \frac{(3\alpha c)^{1/2}}{b}\varepsilon^{1/2},$$

The limit point exists when $\alpha < 0$. A small imperfection may substantially reduce the critical control parameter.

When the perfect structure is symmetric, the corresponding imperfect structure has the free energy

$$G(q, \lambda, \varepsilon) = b(\lambda - \lambda_c)q^2 + dq^4 + \alpha\varepsilon q$$

We adopt the sign convention $b > 0$. When $d > 0$, the load will keep increasing with the displacement, and the structure is said to be imperfection-insensitive. When $d < 0$, the load-displacement will exhibit a maximum load. If the actual load exceeds this maximum, the structure will snap to a distant state. The structure is said to be imperfection-sensitive.
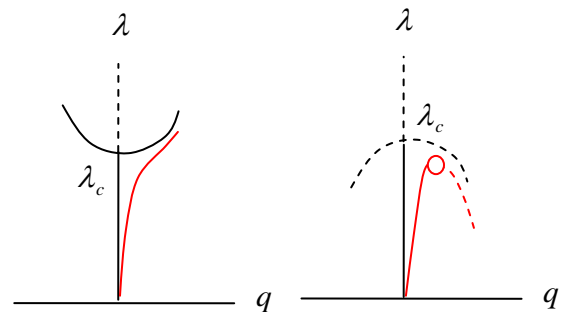
Setting $\partial G / \partial q = 0$ and $\partial^2 G / \partial q^2 = 0$, we obtain that

$$2b(\lambda - \lambda_c)q + 4dq^3 + \alpha\varepsilon = 0.$$
$$2b(\lambda - \lambda_c) + 12dq^2 = 0$$

The combination of the above gives the location of the critical point of the family of imperfect systems:

$$q_s = \left(\frac{\alpha\varepsilon}{8d}\right)^{1/3}, \quad \lambda_s = \lambda_c + \frac{3d^{1/3}}{2b}(\alpha\varepsilon)^{2/3}.$$

Once again, a small imperfection can make the snapping load λ_s much lower than the critical load λ_c of the perfect structure.

Example. We now apply the theory to the example given at the beginning. We first express the energy G of the system as a function of the generalized coordinate, θ. The energy consists of the potential energy of the weight and the potential energy of the spring:

$$G = WL(\cos\theta - 1) + \frac{1}{2}K_2\theta^2 + \frac{1}{3}K_3\theta^3 + \frac{1}{4}K_4\theta^4.$$

Guided by the above theory, we have expressed the elastic energy of the spring in a power series up to fourth order. In this case, $\theta = 0$ is an equilibrium configuration for all value of the weight *W*.

Recall the Taylor expansion

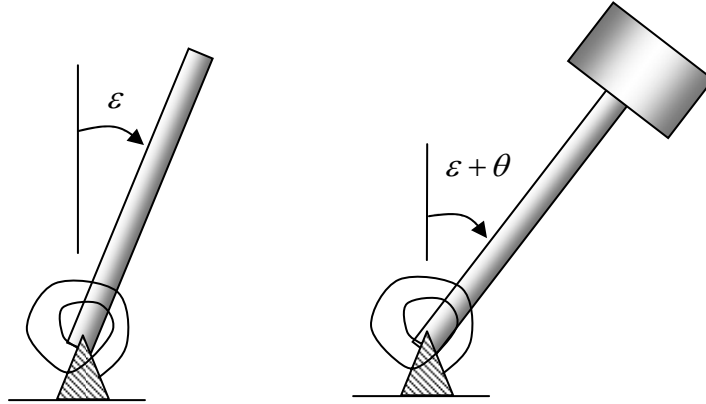$$\cos\theta = 1 - \theta^2 + \frac{\theta^4}{24} - \dots$$

We obtain the Taylor expansion for the energy function:

$$G = \frac{1}{2}(K_2 - WL)\theta^2 + \frac{1}{3}K_3\theta^3 + \left(\frac{1}{4}K_4 + \frac{PL}{24}\right)\theta^4$$

Consequently, the critical weight is

$$W_c = \frac{K_2}{L}.$$

The critical weight W_c is large when the spring is stiff or when the rod is short. The discussion of the behavior around the critical point is straightforward.

One way to introduce an imperfection is to assume that in its unloaded state, the spring is installed such that the rod has some initial deviation ε from the vertical position. Thus, the energy function becomes

$$G = WL[\cos(\theta + \varepsilon) - \cos\varepsilon] + \frac{1}{2}K_2\theta^2 + \frac{1}{3}K_3\theta^3 + \frac{1}{4}K_4\theta^4$$

The leading-order expansion of the energy function is

$$G = \frac{1}{2}(K_2 - WL)\theta^2 + \frac{1}{3}K_3\theta^3 + \left(\frac{1}{4}K_4 + \frac{PL}{24}\right)\theta^4 - \varepsilon\theta.$$

A family of systems of many degrees of freedom. **Bifurcation analysis**. Describe the configuration of a system by a set of n generalized coordinates $q_1,...,q_n$. Describe a family of such systems by varying a control parameter λ. The family of the systems has the free-energy function $G(q_1,...,q_n,\lambda)$. We will be dealing with family of systems with a smooth free-energy function, and with configurational transitions of the second type.

The basic principle remains the same. Given a member system with a fixed λ, the system is in a stable equilibrium configuration when $q_1,...,q_n$ minimizes G. We next establish the conditions of equilibrium and stability.

When the configuration of a member system with a fixed λ changes slightly from q_i to $q_i + \delta q_i$, the free energy of the system changes by

$$G(q_1 + \delta q_1, ..., q_n + \delta q_n, \lambda) - G(q_1, ..., q_n, \lambda) = \frac{\partial G(q_1, ..., q_n, \lambda)}{\partial q_i} \delta q_i + \frac{1}{2} \frac{\partial^2 G(q_1, ..., q_n, \lambda)}{\partial q_i \partial q_j} \delta q_i \delta q_j .$$

A configuration q_i is in equilibrium if

$$\frac{\partial G(q_1, ..., q_n, \lambda)}{\partial q_i} = 0 .$$

This is a set of *n* equilibrium equations. This set of equations determines all stationary configurations: corresponding to minima, maxima and saddle points on the free-energy surface.

For the equilibrium configuration to be stable, it must be a minimum of *G*. That is, for any small change δq in the configuration, the second variation of the energy must be positive. That is, the inequality

$$\frac{\partial^2 G(q_1, ..., q_n, \lambda)}{\partial q_i \partial q_j} \delta q_i \delta q_j > 0$$

holds true for all combinations of δq_i (except for $\delta q_1 = ... = \delta q_n = 0$).

The matrix

$$H_{ij}(\lambda) = \frac{\partial^2 G(q_1, ..., q_n, \lambda)}{\partial q_i \partial q_j}$$

is known as the Hessian or the Hessian matrix. The matrix is also known as the stability matrix or stiffness matrix. All the derivatives are evaluated at an equilibrium configuration, so that the Hessian is a matrix associated with an equilibrium configuration. For a given equilibrium configuration, the energy is a minimum if the Hessian is positive-definite, the energy is a maximum if the Hessian is negative-definite, and the energy is a saddle point if the Hessian is indefinite.

All the above considerations pertain to a member system with a fixed λ. We now vary λ to study the family of systems as a whole. As λ varies, a minimum may become a non-minimum. That is, the Hessian makes a transition from a positive-definite matrix to an indefinite or even a negative-definite matrix. According to linear algebra, this transition occurs when

$$\det(H_{ij}(\lambda)) = 0 .$$

This equation, in conjunction with the *n* equilibrium equations, determines the critical point (λ^c, q_i^c).

Mode of bifurcation. We can gain insight into the critical point by asking another question: Given a member system of a fixed λ, in the neighborhood of an equilibrium configuration q_j^E, can we find another equilibrium configuration? Let a configuration in the neighborhood of the equilibrium configuration be

$$q_j = q_j^E + \delta q_j .$$

For q_j to be an equilibrium configuration, the *n* coordinates must satisfy the equilibrium equations:

$$\frac{\partial G(q_1, ..., q_n, \lambda)}{\partial q_i} = 0 .$$

Expanding these equations into a Taylor series around the given equilibrium configuration q_j^E, we obtain that

$$H_{ij}(\lambda)\delta q_j = 0.$$

The Hessian is evaluated at q_j^E. The process is the same as linearize the equilibrium equations around the given equilibrium configuration.

This defines an eigenvalue problem. According to linear algebra, for the above equation to have a nontrivial solution δq_j, we must set $\det(H_{ij}(\lambda)) = 0$. Once the critical control parameter is solved, we can determine an engenvector δq_j up to a scalar. The combination of the eigenvalue and the eigenvector describes a mode of bifurcation. In general, the equation $\det(H_{ij}(\lambda)) = 0$ may have multiple roots, so that multiple modes of bifurcation may exist.

Vibration in the neighborhood of an equilibrium configuration. Further insight may be gained by considering vibration in the neighborhood of an equilibrium configuration q_j^E. Let a neighboring configuration be

$$q_j(t) = q_j^E + \delta q_j(t)$$

We now interpret $-\partial G / \partial q_j$ as the generalized force applied to the structure. Momentum balance leads to n equations of motion:

$$M_{ij}\delta \ddot{q}_j + H_{ij}\delta q_j = 0.$$

Both matrices M and H may depend on λ, but are independent of δq_j.

The equations of motion determine the perturbation $\delta q(t)$ by an initial value problem. The solution has the form

$$\delta q_j(t) = a_j e^{i\omega t},$$

where $a_j = \delta q_j(0)$. Inserting this form into the equations of motion, we obtain

$$\omega^2 M_{ij} a_j = H_{ij} a_j.$$

The mass matrix is positive-definite. When the equilibrium configuration corresponds to an energy minimum, the Hessian is positive-definite, and the natural frequency is a real number. When $\lambda \to \lambda_c$ the frequency vanishes. When $\lambda > \lambda_c$, the frequency becomes imaginary, which means that $q(t)$ amplifies exponentially. Once again, this leads to the eigenvalue problem $H_{ij}(\lambda)\delta q_j = 0$.

A model of two degrees of freedom. A system consists of one weight, two rigid rods, and two torsional springs. The system has 2 degrees of freedom: θ_1 and θ_2. The free energy of the system is

$$G = WL(\cos\theta_1 + \cos\theta_2 - 2) + \frac{1}{2}K\theta_1^2 + \frac{1}{2}K(\theta_2 - \theta_1)^2.$$

We have only included the quadratic terms for the elastic energy of the springs; higher order terms may be added if necessary. Setting $\partial G / \partial \theta_1 = \partial G / \partial \theta_2 = 0$, we obtain the two equilibrium equations:

$$WL\sin\theta_1 = K\theta_1 - K(\theta_2 - \theta_1),$$

$$WL\sin\theta_2 = K(\theta_2 - \theta_1).$$
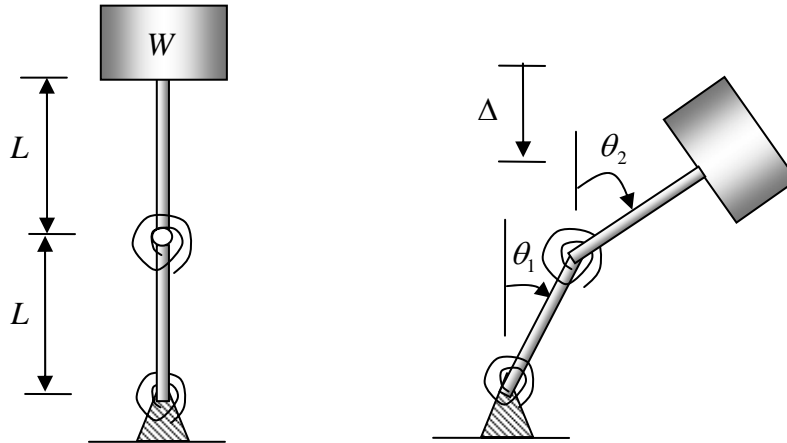
Introduce a dimensionless control parameter

$$\lambda = \frac{WL}{K}.$$

Linearize the equilibrium equations, and we obtain that

$$\lambda\theta_1 = 2\theta_1 - \theta_2$$

$$\lambda\theta_2 = \theta_2 - \theta_1$$

Write the equations in a matrix form:

$$\begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As expected, the Hessian is a symmetric matrix. This is a set of homogenous equations for θ_1 and θ_2. To have a nontrivial solution, the determinant of the matrix must vanish, giving

$$(\lambda - 1)(\lambda - 2) - 1 = 0.$$

This quadratic equation has two solutions:

$$\lambda_I = \frac{3 - \sqrt{5}}{2} \approx 0.38, \quad \lambda_{II} = \frac{3 + \sqrt{5}}{2} \approx 2.62.$$

These are the eigenvalues.

The linearized equations cannot determine the magnitude of the distortion, but can determine the ratios of the two angles:

$$\theta_1 / \theta_2 = 1 - \lambda .$$

For mode I, $\theta_1 / \theta_2 = 0.62$. For mode II, $\theta_1 / \theta_2 = -1.62$.

Elastica: bending of an inextensible rod. In your first undergraduate course on solid mechanics, you have studied stretching, bending, twisting and bucking of a slender rod. In such a course, you typically assume that shape of the rod changes slightly. This assumption enables you to derive many useful results quickly, but obscures some basic ideas, e.g., the nature of buckling and what happens after buckling. Of course, such a careless theory will give wrong predictions when shape change is large. We now formulate a theory of rod undergoing large shape change.

Playing with a long rod, we find that the rod is easy to bend, but hard to stretch. To simplify the matter, we assume that the rod is inextensible. A slender rod undergoing large deflection but no elongation is known as an *elastica*. We start with the 3 ingredients of solid mechanics.

Deformation geometry. To simplify the matter further, we restrict ourselves to an elastica bending within a plane. Each point in this plane is described by a pair of coordinates *x* and *y*. In its "natural state", when no external load is applied, the elastica need not be straight. To name each material particle along the elastica, let *X* be the arc distance between a material particle and one end of the elastica. For each material particle, this length is invariant as the rod deflects, because the rod is taken to be inextensible. At time *t*, the particle *X* moves to a point with coordinates $x(X,t)$ and $y(X,t)$. The two functions describe a curve, i.e., the state of the elastica at time *t*. To evolve the state of the elastica, we next formulate the equations of motion.

In the current state, let $\theta(X,t)$ be the slope (i.e., the angle from the *x*-axis to the tangent direction of the rod), and $\kappa(X,t)$ be the curvature (i.e., the inverse of the radius when the segment is fitted to a part of a circle). The elementary differential geometry of a curve dictates that

$$\frac{\partial x(X,t)}{\partial X} = \cos \theta ,$$

$$\frac{\partial y(X,t)}{\partial X} = \sin \theta ,$$

$$\frac{\partial \theta(X,t)}{\partial X} = \kappa .$$

We have adopted the sign convention that the curvature is positive when the rod is convex downward.
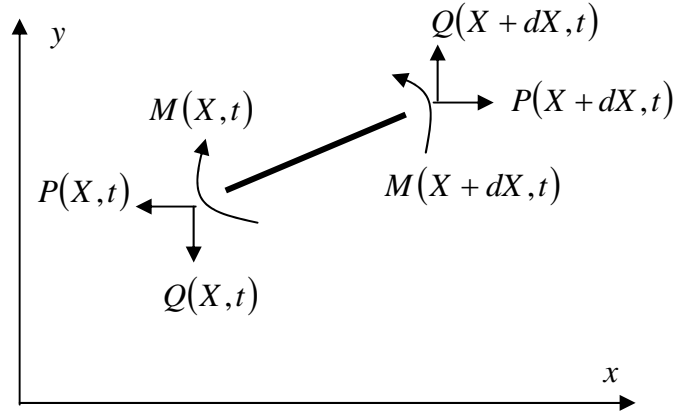
Momentum balance. If we make an imaginary cut at a cross section of the elastica, we expose an internal moment $M(X,t)$, and a pair of internal forces $P(X,t)$ and $Q(X,t)$ along the directions of the two axes. You may project the forces along, and normal to, the tangent direction, and call them axial and shearing forces, but we will not do so here.

We apply Newton's law to an element of the elastic in the current state. Once again, let $\mu(X)$ be the mass of the element divided by the length of the element. The balance of linear momentum along the two axes requires that

$$\frac{\partial P(X,t)}{\partial X} = \mu(X)\frac{\partial^2 x(X,t)}{\partial t^2}$$

$$\frac{\partial Q(X,t)}{\partial X} = \mu(X)\frac{\partial^2 y(X,t)}{\partial t^2}$$

The balance of moment requires that

$$\frac{\partial M(X,t)}{\partial X} = P\sin\theta - Q\cos\theta .$$

We have neglected the effect of inertia associated with rotation.

Material law. The above 6 equations connect 7 functions: 4 geometric quantities and 3 force-like quantities. We need one more equation. This is supplied by a material law connecting the bending moment and curvature. When the curvature is small, bending is elastic, and the bending moment is linear in curvature:

$$M = D\kappa$$

where D is called the bending stiffness, and can be measured experimentally or calculated from a model at the scale of individual cross section of the rod; see a note at the end. For now, we simply regard D as an input parameter to the model at the scale of entire elastica. D has the dimension of force times length squared.

An initially straight rod subject to a pair of forces. Euler buckling. Consider a rod subject to a pair of forces F. If the forces pull the rod, because our model assumes that the rod is inextensible, the model will produce nothing interesting. If the forces push the rod, our experience shows that the rod will undergo a series of states as illustrated in the figure. When the force is small, the rod is straight. When the force exceeds a critical value, the rod buckles. After buckling, the rod continues to bend further.

We assume that the process of deflection is so slow that inertia plays no role. Thus, the two equations of linear momentum balance becomes

$$Q = 0 \text{ and } P = -F .$$

The equation of moment balance becomes that

$$\frac{\partial \kappa(X)}{\partial X} = -\frac{F}{D}\sin\theta$$

This equation, together with basic equations in differential geometry,

$$\frac{\partial x(X)}{\partial X} = \cos\theta,$$

$$\frac{\partial y(X)}{\partial X} = \sin\theta,$$

$$\frac{\partial \theta(X)}{\partial X} = \kappa,$$

Form a set of nonlinear ODEs, with the boundary conditions

$$x(0) = y(0) = 0, \ \kappa(0) = 0$$
$$y(L) = 0, \ \kappa(L) = 0.$$

This boundary value problem can be solved numerically by the shooting method.

Clearly, a straight rod is a trivial solution for all values of the applied force. We next look for a nontrivial solution in the neighborhood of this trivial solution. Following the general procedure, we linearize the equilibrium equations around the trivial equilibrium configuration. When the rod is still nearly straight, the slope is small, $\theta \ll 1$. Recall the Taylor series to the first power in θ:

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2}.$$

The four ODEs become

$$x = X, \ \frac{\partial y}{\partial x} = \theta, \ \frac{\partial \theta}{\partial x} = \kappa, \ \frac{\partial \kappa}{\partial x} + \frac{F}{D}\theta = 0.$$

The function

$$y = \sin(\pi X / L)$$

satisfies the equilibrium equation and the boundary conditions, provided that the axial force is given by

$$F_c = \pi^2 \frac{D}{L^2}.$$

This is the Euler condition for bucking.

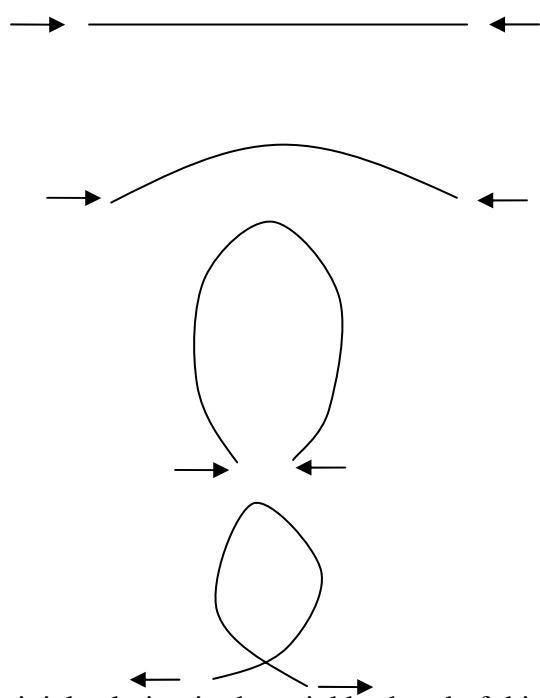**A rod under a pair of forces and undergoing transverse vibration.** If the rod vibrates in the transverse direction, the equation of motion for deflection $y(X,t)$ is

$$D\frac{\partial^4 y}{\partial X^4} + F\frac{\partial^2 y}{\partial X^2} = -\mu\frac{\partial^2 y}{\partial t^2}.$$

Try the solution of the form

$$y(X,t) = \sin\left(\frac{\pi X}{L}\right)\sin\omega t.$$

We obtain the frequency

$$\omega = \pi^2 \sqrt{\frac{D}{\mu L^4}\left(1 - \frac{F}{F_c}\right)}.$$

Consequently, the frequency decreases when the compressive force increases, and approaches zero when the axial force approaches the Euler condition.

Demonstrate the effect of axial force on frequency in class using a column with a variable length.

Finite elastic deformation. A more detailed formulation of finite deformation is available at http://imechanica.org/node/538.

Deformation geometry. A body consists of a field of material particles. We name a *particle* by its coordinates $\mathbf{X}$ in the reference state, and name the *place* of the particle at time t by the coordinates $\mathbf{x}$ of the particle in the current state. The function $\mathbf{x}(\mathbf{X},t)$ describes the history of the deformation of the body. The object of the continuum mechanics is to formulate algorithms to evolve the function $\mathbf{x}(\mathbf{X},t)$.

For a tensile bar, the stretch is defined by the length of the bar in the current state divided by the length of the bar in the reference state, $\lambda = l/L$. A generalization of this definition to inhomogeneous deformation of the three-dimensional body is the *deformation gradient tensor*:

$$F_{iK}(\mathbf{X},t) = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K}.$$

In addition to describe stretching, however, the deformation tensor also describes rigid-body rotation. To eliminate the rigid-body rotation, we can invoke the Lagrange strain tensor**.** A material element of line is $d\mathbf{X}$ in the reference state, and is $d\mathbf{x}$ in the current state. The length of the element in the reference state, dL, is given by

$$dL^2 = dX_K dX_K.$$

The length of the same element in the current state, dl, is given by

$$dl^2 = dx_i dx_i.$$

Define the Lagrange strain tensor E_{KL} by

$$dx_i dx_i - dX_K dX_K = 2E_{KL}dX_K dX_L.$$

We can express this tensor in terms of the deformation gradient:

$$E_{KL} = \frac{1}{2}\left(F_{iK}F_{iL} - \delta_{KL}\right),$$

where $\delta_{KL} = 1$ when $K = L$, and $\delta_{KL} = 0$ when $K \neq L$. The E_{KL} tensor is symmetric.

Momentum balance. Let $dV(\mathbf{X})$ be an element of material volume in the reference state, and let $\mathbf{B}(\mathbf{X},t)dV(\mathbf{X})$ be the force in the current state acting on the material element of volume. Let $\mathbf{N}(\mathbf{X})dA(\mathbf{X})$ be an element of material area in the reference state, and let $\mathbf{T}(\mathbf{X},t)dA(\mathbf{X})$ be the force in the current state acting on the material element of area. Let $\rho(\mathbf{X})dV(\mathbf{X})$ be the mass of the material element of volume. Because the mass of the material element is invariant during deformation, $\rho(\mathbf{X})$ is time-independent. Consider a material element of area normal to the axis X_K when the element is in the reference state. Thus, s_{iK} is component *i* of the force acting on the element in the current state divided by the area of the element in the reference state. The tensor $s_{iK}(\mathbf{X},t)$ is called the nominal stress.
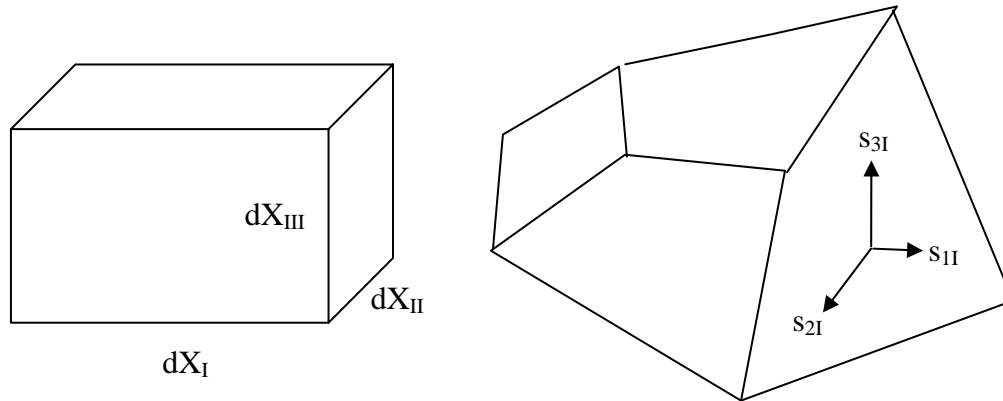
The balance of momentum requires that the nominal stress obey

$$\frac{\partial s_{iK}(\mathbf{X},t)}{\partial X_K} + B_i(\mathbf{X},t) = \rho(\mathbf{X})\frac{\partial^2 x_i(\mathbf{X},t)}{\partial t^2},$$

in the volume of the body, and

$$s_{iK}N_K = T_i$$

on the surface of the body.

Hyperelastic material.   Let $WdV(\mathbf{X})$ be the elastic energy stored in an element of volume in the current state. When the deformation gradient changes by δF_{iK}, the stress does work $s_{iK}\delta F_{iK}$. Assume that all this work is stored as the elastic energy in the particle, so that

$$dW = s_{iK}\delta F_{iK}.$$

Consequently, the energy is a function of the tensor of deformation gradient, $W(\mathbf{F})$. Once this function is prescribed, the material law is

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}}.$$

The deformation gradient $\mathbf{F}$ consists of both stretch and rotation. The energy is invariant if the particle undergoes a rigid body rotation. To exclude rigid-body rotation, we can take the energy to be a function of the Lagrange strain, $W(\mathbf{E})$. The nominal stress is

$$s_{iK} = \frac{\partial W(\mathbf{E})}{\partial E_{MN}}\frac{\partial E_{MN}}{\partial F_{iK}},$$

or

$$s_{iK} = F_{iL}\frac{\partial W(\mathbf{E})}{\partial E_{LK}}.$$

Perturbations around a given equilibrium state.   Let $\mathbf{x}(\mathbf{X})$, $\mathbf{F}(\mathbf{X})$ and $\mathbf{s}(\mathbf{X})$ be the quantities associated with a given equilibrium state, They are the solutions of a static boundary value problem:

$$F_{iK}(\mathbf{X}) = \frac{\partial x_i(\mathbf{X})}{\partial X_K},$$

$$\frac{\partial s_{iK}(\mathbf{X})}{\partial X_K} + B_i(\mathbf{X}) = 0 \,,$$

$$s_{iK} = \frac{\partial W(\mathbf{F})}{\partial F_{iK}} \,,$$

On any part of the surface of the body, either $\mathbf{x}$ or $s_{iK}N_K$ is prescribed. Also prescribed are the body force $\mathbf{B}$ and the free-energy function W.

We ask the following questions:

1. *The question of vibration.* In the neighborhood of the given equilibrium state, how does the body vibrate?
2. *The question of bifurcation.* In the neighborhood of the given equilibrium state, can we find another equilibrium state?

Let $\mathbf{x}(\mathbf{X}) + \delta\mathbf{x}(\mathbf{X},t)$, $\mathbf{F}(\mathbf{X}) + \delta\mathbf{F}(\mathbf{X},t)$ and $\mathbf{s}(\mathbf{X}) + \delta\mathbf{s}(\mathbf{X},t)$ be the quantities associated with a deformation in the neighborhood of the given equilibrium state. The perturbations satisfy the following equations:

$$\delta F_{iK} = \frac{\partial \delta x_i(\mathbf{X},t)}{\partial X_K} \,,$$

$$\frac{\partial \delta s_{iK}}{\partial X_K} = \rho \frac{\partial^2 \delta x_i}{\partial t^2} \,,$$

$$\delta s_{iK} = K_{iKjL}\,\delta F_{jL} \,.$$

On surface of the body, $\delta\mathbf{x}$ vanishes where displacement is prescribed, and $N_K\delta s_{iK}$ vanishes where traction is prescribed. The tensor of tangent modulus, K_{iKjL}, is calculated from the free-energy function:

$$K_{iKjL}(\mathbf{F}) = \frac{\partial s_{iK}(\mathbf{F})}{\partial F_{jL}} = \frac{\partial^2 W(\mathbf{F})}{\partial F_{jL}\,\partial F_{iK}} \,.$$

When the given equilibrium equation is inhomogeneous, $\mathbf{F}$ varies with $\mathbf{X}$, and the tangent modulus also varies with $\mathbf{X}$.

A combination of the above equations gives

$$\frac{\partial}{\partial X_K}\left(K_{iKjL} \frac{\partial \delta x_j}{\partial X_L} \right) = \rho \frac{\partial^2 \delta x_i}{\partial t^2} \,.$$

This is the equation of motion for $\delta\mathbf{x}(\mathbf{X},t)$. In conjunction with the boundary conditions, this equation defines a linear, homogenous problem.

Vibration. We look for solution of the form

$$\delta x_i(\mathbf{X},t) = a_i(\mathbf{X})e^{i\omega t} \,,$$

where ω is a natural frequency, and $a_i(\mathbf{X})$ is the field of amplitude of the vibration. Inserting into the equation of motion, we obtain that

$$\frac{\partial}{\partial X_K}\left(K_{iKjL} \frac{\partial a_j}{\partial X_L} \right) = -\rho\omega^2 a_i \,.$$

This equation defines an eigenvalue problem, together with the boundary conditions:

- $a_i = 0$ on the part of the surface of the body where the displacement is prescribed
- $N_K K_{iKjL} \partial a_j / \partial X_L = 0$ on the part of the surface of the body where the traction is prescribed

The eigenvalue problem has several usual properties. First, the eigenfunctions associated with the distinct eigenvalues are orthogonal. Let ω' and ω'' be two distinct eigenvalues, and a' and a'' be the associated eigenfunctions. Thus,

$$\frac{\partial}{\partial X_K}\left(K_{iKjL} \frac{\partial a'_j}{\partial X_L}\right) = -\rho \omega'^2 a'_i$$

in the volume of the body. On the surface of the body $a'_i = 0$ where the displacement is prescribed, and $N_K K_{iKjL} \partial a'_j / \partial X_L = 0$ where the traction is prescribed.

We can write similar equations for the other eigenmode. Multiplying the PDE by a''_i and integrating over the volume of the body, we obtain that

$$\int a''_i \frac{\partial}{\partial X_K}\left(K_{iKjL} \frac{\partial a'_j}{\partial X_L}\right) dV = -\rho \omega'^2 \int a''_i a'_i dV .$$

Using divergence theorem, we can write the above equation as

$$\int a''_i N_K \left(K_{iKjL} \frac{\partial a'_j}{\partial X_L}\right) dA - \int K_{iKjL} \frac{\partial a''_i}{\partial X_K} \frac{\partial a'_j}{\partial X_L} dV = -\rho \omega'^2 \int a''_i a'_i dV .$$

The first integral extends to the surface of the body, and vanishes due to the boundary conditions, so that

$$\int K_{iKjL} \frac{\partial a''_i}{\partial X_K} \frac{\partial a'_j}{\partial X_L} dV = \rho \omega'^2 \int a''_i a'_i dV .$$

Exchanging the roles of the two modes, we can similarly write

$$\int K_{iKjL} \frac{\partial a'_i}{\partial X_K} \frac{\partial a''_j}{\partial X_L} dV = \rho \omega''^2 \int a''_i a'_i dV .$$

Note that

$$K_{iKjL} = K_{jLiK} ,$$

So that the integrals on the left side of the two equations are identical. We conclude that

$$\int a''_i a'_i dV = 0 .$$

That is, the eigenfunctions of the two modes are orthogonal.

Second, the eigenvalue ω^2 must be real. Otherwise, if ω^2 were a complex eigenvalue, $\overline{\omega}^2$ would be a different eigenvalue. Let a_i be the eigenfunction associated with ω^2, so that $\overline{a}_i$ would be the eigenfunction associated with $\overline{\omega}^2$. Consequently, $\int \overline{a}_i a_i dV = 0$. This would be possible only if $a_i = 0$.

Bifurcation. When an eigenvalue ω^2 is positive, it corresponds to a mode of vibration. When an eigenvalue ω^2 is negative, it corresponds to a perturbation exponentially growing with time. The critical condition is that the eigenvalue is zero. Thus,

$$\frac{\partial}{\partial X_K}\left(K_{iKjL} \frac{\partial \delta x_j}{\partial X_L}\right) = 0$$

in the volume of the body. On the surface of the body, $\delta x_i = 0$ where the displacement is prescribed, and $N_K K_{iKjL} \partial \delta x_j / \partial X_L = 0$ where the traction is prescribed.

A usual scenario is that the given equilibrium state changes with, say an applied load. As the load increases, the natural frequency decreases. At a critical load, the natural frequency vanishes. This critical load also corresponds to the point of bifurcation where we can find other equilibrium states in the neighborhood of a given equilibrium state.

Perturbation around a homogenous equilibrium state. When the given equilibrium state is homogeneous, namely, when the deformation gradient tensor **F** is the same for all material particles in the body, the tangent modulus will also be the same for all material particles in the body. However, the perturbation can be inhomogeneous, $\delta x_i(\mathbf{X}, t)$. The equation of motion becomes

$$K_{iKjL} \frac{\partial^2 \delta x_j}{\partial X_K \partial X_L} = \rho \frac{\partial^2 \delta x_i}{\partial t^2}.$$

On the surface of the body, $\delta x_i = 0$ where the displacement is prescribed, and $N_K K_{iKjL} \partial \delta x_j / \partial X_L = 0$ where the traction is prescribed.

Vibration. Inspecting the governing equation, we note a quantity of dimension of velocity:

$$\sqrt{\frac{K}{\rho}},$$

where K is a representative magnitude of the tangent modulus. The governing equation has no length scale. However, the size of the body provides one or more length scales. Let L be a length scale provided by the size of the body. The fundamental frequency scales as

$$\omega \sim \frac{1}{L}\sqrt{\frac{K}{\rho}}.$$

The eigenfunction associated with the fundament frequency is inhomogeneous over the scale of L. By contrast, eigenfunctions associated with higher frequencies are inhomogeneous over shorter length scales. When you use finite element method to determine vibration modes, you need a coarse mesh to resolve the fundamental mode, but a fine mesh to resolve a higher mode.

Wave. When the disturbance diffused, namely, is over the length scale comparable to the size of the body, the disturbance may be described by a linear superposition of a few vibration modes of low frequencies. However, when the disturbance is localized, namely, is over a length scale much smaller than the size of the body, vibration modes are often not an effective way to describe such localized disturbance. Rather, we may describe the localized disturbance in terms of waves. For example, consider an infinite medium. Assume a plane wave with unit normal vector **N** and speed c. The disturbance takes the form

$$\delta \mathbf{x}(\mathbf{X}, t) = \delta \mathbf{a} f(\mathbf{N} \cdot \mathbf{X} - ct),$$

where $\delta \mathbf{a}$ is the direction of the displacement, and f is the profile of the wave.

The equation of motion becomes

$$K_{iKjL} N_K N_L \delta a_j = \rho c^2 \delta a_i.$$

This is an eigenvalue problem. When the tangent modulus is positive-definite, we can find three distinct plane waves. In this case, the wave is nondispersive, so that any it can propagation with an arbitrary profile f. See the section on waves in ES 240 (http://imechanica.org/node/385).

We can similarly consider a wave propagating along a surface (the Rayleigh wave), or a wave propagating in a layer (the Love wave).

Bifurcation. If we are looking for an equilibrium state in the neighborhood of the given equilibrium state, the governing equation is

$$K_{iKjL}\frac{\partial^2 \delta x_j}{\partial X_K \partial X_L} = 0.$$

The same homogenous boundary conditions apply. The tensor of tangent modulus depends on the state of homogeneous deformation, or a loading parameter. The loading parameter serves the role of an eigenvalue. We may analyze the diffused bifurcation modes by solving this eigenvalue problem.

As the loading parameter increases, more and more localized bifurcation modes become possible. In the limiting case, we may consider an infinite medium and a bifurcation of the form

$$\delta \mathbf{x}(\mathbf{X},t) = \delta \mathbf{a} f(\mathbf{N} \cdot \mathbf{X}).$$

Inserting into the above equation, we obtain that

$$K_{iKjL} N_K N_L \delta a_j = 0.$$

For a given direction $\mathbf{N}$, this equation defines an eigenvalue problem that determines the critical loading parameter and the associated eigen-displacement $\delta\mathbf{a}$. The solution is interpreted as a wave with a vanishing velocity.

There is a large body of literature along these lines. Truesdell and Noll reviewed works dating back at least 100 years. I'd like to mention a few papers by local authors:

- R. Hill and J.W. Hutchinson, Bifurcation phenomena in the plane tension test, Journal of the Mechanics and Physics of Solids 23, 239-264 (1975).
- J. R. Rice, The localization of plastic deformation, Proceedings of the 14th IUTAM Congress, 1976 (http://esag.harvard.edu/rice/062_Rice_LocalPlasDef_IUTAM76.pdf).
- T. Li, Z. Suo, Deformability of thin metal films on elastomer substrates. International Journal of Solids and Structures 43, 2351-2363 (2006).

Considère reconsidered. A bar is in a homogenous state of uniaxial tension. Derive the equation of motion that evolves infinitesimal disturbance along the bar. Use the equation to examine the Considère condition. We will need to examine an inhomogeneous deformation along the bar. When the variation along the bar is over a length scale larger than the diameter of the bar, we may model the homogenous deformation by assuming a uniaxial stress state.

Let the bar be in the homogenous deformation of stretch λ. Superimposed on the homogenous deformation is an inhomogeneous deformation $\delta x(X,t)$. The increment in the stretch is

$$\delta \lambda = \frac{\partial \delta x(X,t)}{\partial X}$$

Let the increment in the nominal stress be $\delta s(X,t)$. Momentum balance requires that

$$\frac{\partial \delta s(X,t)}{\partial X} = \rho \frac{\partial^2 \delta x(X,t)}{\partial t^2} .$$

We specify the material law by a function $s(\lambda)$. Thus, the tangent modulus K is defined by

$$\delta s = K(\lambda)\delta \lambda$$

Because the unperturbed state is homogeneous, $K(\lambda)$ is independent of X and t.

The equation of motion becomes

$$K(\lambda)\frac{\partial^2 \delta x(X,t)}{\partial X^2} = \rho\frac{\partial^2 \delta x(X,t)}{\partial t^2}.$$

This is the familiar wave equation that evolves the perturbation $\delta x(X,t)$. When $K(\lambda) > 0$, the disturbance can propagate in the bar at velocity

$$c = \sqrt{\frac{K(\lambda)}{\rho}}.$$

When $K(\lambda) = 0$, a time independent inhomogeneous deformation becomes possible.

We can express the tangent modulus in a familiar form. Recall the relation between the nominal stress and strain

$$s(\lambda) = \frac{\sigma(\lambda)}{\lambda}.$$

The tangent modulus is

$$K(\lambda) = \frac{ds(\lambda)}{d\lambda} = \frac{1}{\lambda}\left[\frac{d\sigma(\lambda)}{d\lambda} - \frac{\sigma(\lambda)}{\lambda}\right].$$

The condition $K(\lambda) = 0$ corresponds to

$$\frac{d\sigma(\lambda)}{d\lambda} = \frac{\sigma(\lambda)}{\lambda}.$$

This is the same as the Considère condition.

Luders's band and Maxwell's rule. When a bar of some metals or polymers is subject to a tensile load, deformation may become inhomogeneous along the bar. Discrete bands of the material deform more than the rest of the bar. These bands are known as Luders's bands. As the length of the bar increases, the fronts of the bands move, until the entire bands become more deformed. Although inelastic deformation is usually associated with Luders's bands, here we will model the material as a nonlinearly elastic material.

We take the undeformed state the reference state, where the bar has length L and cross-sectional area A. In the current state, a weight applies force P. First assume that the bar deforms homogeneously to length l and area a. In the current state, when the length of the bar changes by δl, the weight does work $P\delta l$. The volume of the bar in the reference state is AL. Consequently the work done by the weight in the current state divided by the volume of the bar in the reference state is $s\delta\lambda$, where $s = P/A$ is the nominal stress, and $\lambda = l/L$ is the stretch. The nominal stress is work-conjugate to the stretch.

The bar is in thermal contact with a heat bath held at a fixed temperature. Thus, the state of the bar is characterized by one independent variable, the stretch λ. Let $W(\lambda)$ be the free energy of the bar in the current state divided by the volume of the bar in the reference configuration. The free energy of the weight is $-Pl = -LAs\lambda$. Consequently, the free energy of the system, consisting of the bar and the weight, is

$$G = LAW(\lambda) - LAs\lambda.$$

We fix the weight, and vary the stretch λ. Thermodynamics dictates that when the bar equilibrates with the weight, the values of λ should minimize the free energy.

Imagine that the system is in a state of λ, and is perturbed slightly to a state $\lambda + \delta\lambda$. The perturbation changes the free energy by

$$\frac{\delta G}{LA} = \left(\frac{\partial W}{\partial \lambda} - s\right)\delta\lambda + \frac{\partial^2 W}{2\partial\lambda^2}(\delta\lambda)^2.$$

This is the Taylor expansion up to the second order in $\delta\lambda$. For G to reach a local minimum, the first-order variation must vanish:

$$s = \frac{\partial W(\lambda)}{\partial \lambda}.$$

and the second-order variation must be positive-definite:

$$\frac{\partial^2 W}{\partial \lambda^2} > 0.$$

The equilibrium condition is anticipated because the nominal stress is work-conjugate to the stretch. The stability condition also has a simple interpretation: for a state to be stable, the energy function must be convex.

Now consider a material whose free-energy function is non-convex. As sketched in the Figure, the function is convex for small and large values of the stretch, but is concave for intermediate values of stretch.

We can also sketch the free energy of the composite system of the bar and the weight, G. Each curve corresponds to a nominal stress. For a small or a large stress, the free energy function has a single minimum, corresponding to a stable equilibrium state. For an intermediate range of stress, the function has two minima, with the lower one corresponding to a stable equilibrium state, and the higher one corresponding to a metastable equilibrium state. At a particular nominal stress, s^*, the two minima have the equal value of the free energy. The significance of s^* is understood as follows. Suppose that the state of the bar is no longer homogenous, but is composed of two states. The material of the two states occupies lengths L' and L''. In this simplified treatment, we will neglect the transition region in the bar between the two states, so that the total length in the reference state is $L' + L'' = L$. The deformed length of the bar is $L'\lambda' + L''\lambda'' = l$, and the and the free energy of the composite system of the bar and the weight is

$$G = L'AW(\lambda') + L'AW(\lambda'') - sA(L' + L'').$$

Thermodynamics requires that in equilibrium this free energy be minimized subject to the constraint $L' + L'' = L$. Setting $\partial G / \partial \lambda' = \partial G / \partial \lambda'' = 0$ and $\partial G / \partial L' = 0$, we obtain that
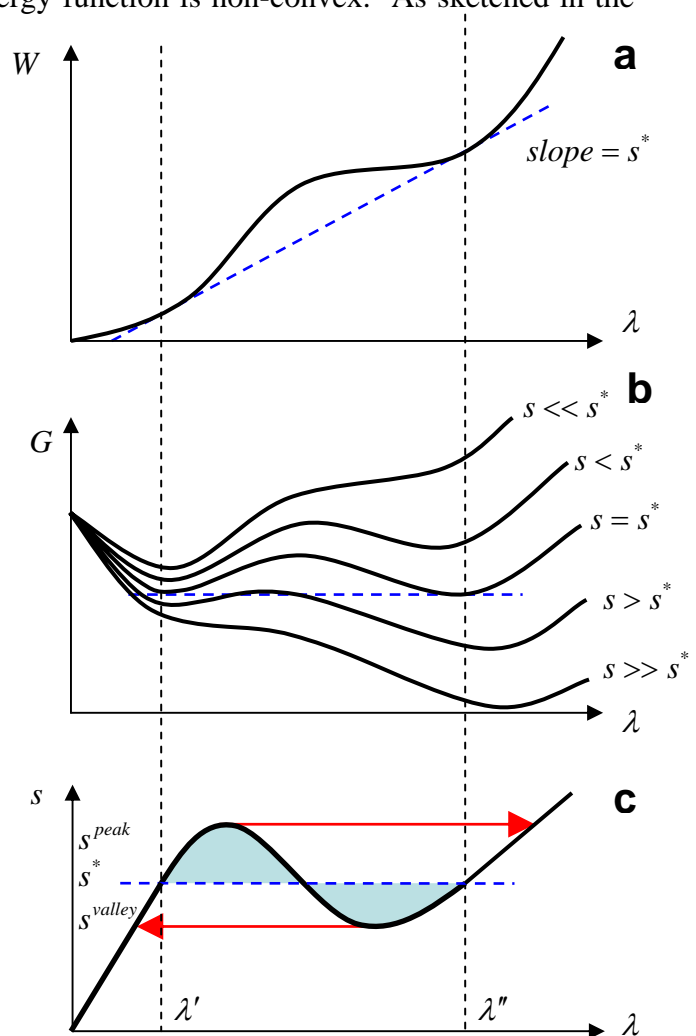
$$s^* = \frac{dW}{d\lambda'} = \frac{dW}{d\lambda''} = \frac{W(\lambda'') - W(\lambda')}{\lambda'' - \lambda'}.$$

These conditions have the familiar graphical interpretations. The two states equilibrate when they lie on the common tangent line of $W(\lambda)$ or, equivalently, when the two minima of G have the same height. The slope of the common tangent gives s^*, the nominal stress under which the two states coexist in equilibrium.

Because the free-energy function is non-convex, its derivative $s = \partial W(\lambda)/\partial \lambda$ is not monotonic. The nominal stress under which the two states coexist in equilibrium, s^*, is at the level such that the two shaded regions have the same area. This interpretation is known as Maxwell's rule. As the length of bar l increases, the length of one state increases as the expense of that of the other.

Coexistent states occur in many phenomena, such as the propagation of bulges along a cylindrical party balloon and buckles along a pipe. For a more recent example, see Xuanhe Zhao, Wei Hong and Zhigang Suo, Electromechanical hysteresis and coexistent states in dielectric elastomers (http://imechanica.org/node/1283).

Ferroelectricity. A crystal has the following property. At a high temperature, the center of the negative change coincide with that of the positive charge, so that the crystal is nonpolar. At a low temperature, however, the crystal is polar. This behavior can be described by a free-energy function

$$W = a(T - T_c)D^2 + bD^4,$$

where $a > 0$ and $b > 0$. We regard the electric displacement D as the generalized coordinate, and the temperature as the control parameter. The behavior of this free energy function has been discussed before. When $T > T_c$, the function has a single minimum at $D = 0$. When $T < T_c$, the state $D = 0$ becomes an energy maximum, and two energy minima occur at

$$D = \pm \sqrt{\frac{a}{2b}(T_c - T)}.$$

The coefficients a and b have been determined experimentally for some crystals.

The above is the basic thermodynamic theory of ferroelectricity. To refine the theory, one may go in several directions. For example, one can account for the symmetry of a crystal. Suppose the crystal has a cubic symmetry at high temperature. A suitable form of the free-energy function is

$$W = a(T - T_c)\left(D_1^2 + D_2^2 + D_3^2\right) + b_{11}\left(D_1^4 + D_2^4 + D_3^4\right) + b_{12}\left(D_1^2 D_2^2 + D_2^2 D_3^2 + D_3^2 D_1^2\right).$$

As another example, it is observed that the crystal changes shape when polarized. This effect may be modeled by a free energy function of the form

$$W = a(T - T_c)D^2 + bD^4 + Y\gamma^2 + gD^2\gamma.$$

The material laws are given by

$$E = \frac{\partial W(D, \gamma)}{\partial D}, \quad \sigma = \frac{\partial W(D, \gamma)}{\partial \gamma}.$$

One can of course extend this into a multiaxial form. See N.A. Pertsev, A.G. Zembilgotov and A.K. Tagantsev, Effect of mechanical boundary conditions on phase diagrams of epitaxial ferroelectric thin films. Physical Review Letters 80, 1988 (1998).