Mixed Mode Fracture

Readings:
J.W. Hutchinson and Z. Suo, "Mixed-mode cracking in layered materials", *Advances in Applied Mechanics*, **29**, 63-191 (1992). (http://www.deas.harvard.edu/suo/papers/17.pdf)

A stationary, planar crack subject to mixed mode loading. First examine a stationary, planar crack in a homogenous, isotropic material. The body is subject to a distribution of load that can both open and shear the crack. Let (r,θ) be the polar coordinates centered at a given point on the crack front. We will consider the situation that the shear load is in the plane (r,θ). The stress field near the crack front has the form

$$\sigma_{ij}(r,\theta)=\frac{K_I}{\sqrt{2\pi r}}\Sigma_{ij}^{I}(\theta)+\frac{K_{II}}{\sqrt{2\pi r}}\Sigma_{ij}^{II}(\theta).$$

The θ-dependent functions, $\Sigma_{ij}^{I}(\theta)$ and $\Sigma_{ij}^{II}(\theta)$, can be found in Lawn (1993). Mode I field is symmetric with the crack plane, with $\Sigma_{22}^{I}(0)=1$ and $\Sigma_{12}^{I}(0)=0$. Mode II field is antisymmetric with the crack plane, with $\Sigma_{22}^{II}(0)=0$ and $\Sigma_{12}^{II}(0)=1$. In particular, the stresses on the plane $\theta=0$ a distance r ahead the crack front are

$$\sigma_{22}(r,0)=\frac{K_I}{\sqrt{2\pi r}},\quad \sigma_{12}(r,0)=\frac{K_{II}}{\sqrt{2\pi r}}.$$

The stress intensity factors, K_I and K_{II}, scale the amplitudes of opening and shearing loads, respectively. We will be interested in situations that the crack front is under tension, rather than compression, namely, $K_I \ge 0$.

Values of stress intensity factors. Once a crack configuration is prescribed, the stress intensity factors K_I and K_{II} are determined by solving the elasticity boundary value problem. For example, for a crack of length 2a in an infinite sheet, subject to remote tensile stress σ and shear stress τ, the two stress intensity factors are

$$K_I=\sigma\sqrt{\pi a} \text{ and } K_{II}=\tau\sqrt{\pi a}.$$

Solutions for many crack configurations have been collected in handbooks. Finite element methods have been used routinely to determine the stress intensity factors under the mixed mode conditions (F.Z. Li, C.F. Shih, A. Needleman. A comparison of methods for calculating energy release rates. *Engineering Fracture Mechanics*. 1985, **21**, 405-42).

Mode mix and amplitude of the load. Again the above stress field is valid in an annulus, with the inner radius larger than the zone of the bond-breaking process, and the outer radius smaller than a length representative of the external boundary conditions. The mixed mode conditions are characterized by two loading parameters, K_I and K_{II}. The two "messengers" transmit the boundary conditions to the crack tip process. The relative amount of mode II to mode I is specified by the mode angle ψ defined by

$$\tan\psi = K_{II}/K_I.$$

With $K_I \ge 0$, the mode angle ranges between $-\pi/2 \le \psi \le \pi/2$. A pure mode I crack corresponds to $\psi=0$, while a pure mode II crack corresponds to either $\psi=+\pi/2$ or $\psi=-\pi/2$.

The energy release rate G is still defined as the elastic energy reduction as the crack extends unit area, while the load does no additional work. The energy release rate relates to the two stress intensity factors by

$$G = \frac{1}{\bar{E}}\left(K_I^2 + K_{II}^2\right).$$

Instead of using K_I and K_{II} to represent the external loads, we use G to represent the amplitude of the loads, and ψ to represent the mode of the loads.

Crack kinking

F. Erdogan and G.C. Sih, On the crack extension in plates under plane loading and transverse shear. *Journal of Basic Engineering, Transactions of the ASME*, Vol. 85, 519-527 (1963).

Under the mixed mode conditions, upon growing, the planar crack often kinks at an angle from its original plane. The kink angle depends on the relative amount of mode II to mode I load. Erdogan and Sih (1963) showed that the experimentally measured kink angles in a plexiglass are well predicted by the criterion that the crack kinks to the plane with the maximum hoop stress. For the stationary crack, the hoop stress near the crack tip is

$$\sigma_{\theta\theta}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}}\cos^3\left(\frac{\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}3\cos^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right).$$

The hoop stress maximizes at an angle θ^*, given by

$$\tan\left(\frac{\theta^*}{2}\right) = -\frac{2\tan\psi}{1+\sqrt{1+8\tan^2\psi}}.$$

When $K_{II} > 0$, $\psi > 0$ and the crack kinks down (i.e., $\theta^* < 0$). The converse is true when $K_{II} < 0$. When the plane crack is pure mode I, this criterion predicts that the crack extends straight ahead. When the plane crack is pure mode II, this criterion predicts that the crack kinks at an angle $\theta^* = 70.5°$.

A moving crack selects a mode I path

M.D. Thouless, A.G. Evans, M.F. Ashby, J.W. Hutchinson. The edge cracking and spalling of brittle plates. *Acta Met.* 1987, **35**, 1333-1341.

The moving crack tip seeks a trajectory that is locally mode I; any nonzero K_{II} will cause the crack to deflect. To illustrate this point, consider two examples. It is well known that a crack in a double-cantilever beam is unstable: the crack tends to curve one way or the other. By symmetry, the crack on the mid-plane of the sample is pure mode I. This mode I path, however, is unstable. A crack, lying slightly off the mid-plane, has a mode II component that tends to deflect the crack further away from the mid-plane.

As a second example, consider a thin film spalling from an adherent substrate. The film is well bonded to the substrate, and is under a residual tensile stress. It is sometimes observed that a crack starts from the edge of the film, dives into the substrate, and then grows parallel to the interface. In the crack wake, the residual stress in the film is partially relieved, and the film and a thin layer of the substrate material underneath form a composite plate, bending up. This observation is peculiar in that, were the crack to run on the film-substrate interface, the residual stress in the film would be fully relieved in the crack wake. Such a global energy consideration has no physical basis: it is the local process of bond breaking that selects the crack path. The experimental observation has been interpreted that the crack tip moves along a mode I trajectory. To see how this works, consider a special case that the thin film and the substrate have similar elastic modulus. Let *h* be the film thickness, and *d* be the depth of the crack parallel to the interface. When the crack is long compared to *d*, the effect of the residual stress on the crack is well described by the equivalent axial force and bending moment:

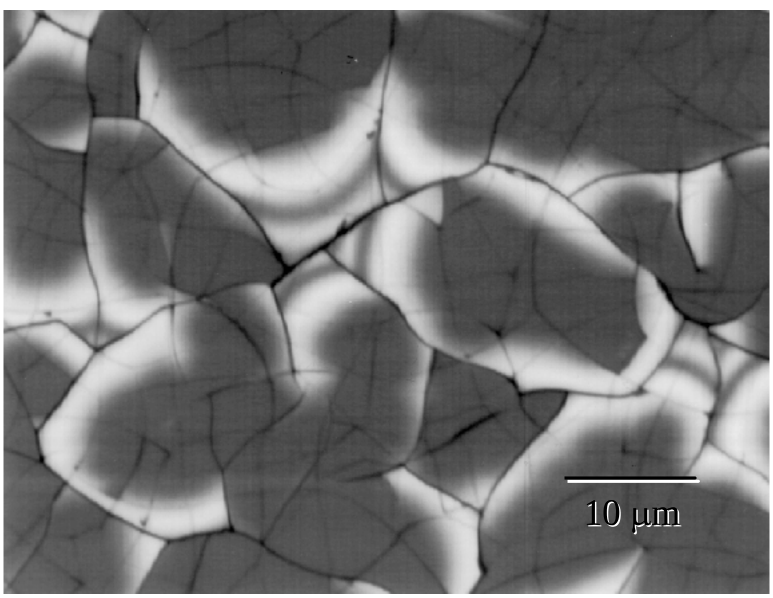
$$P = \sigma h, \quad M = \frac{1}{2}\sigma h(d - h).$$

The stress intensity factors have been calculated, given by

$$K_I = \frac{1}{\sqrt{2}}\left(Pd^{-1/2}\cos\omega + 2\sqrt{3}Md^{-3/2}\sin\omega\right),$$
$$K_{II} = \frac{1}{\sqrt{2}}\left(Pd^{-1/2}\sin\omega - 2\sqrt{3}Md^{-3/2}\cos\omega\right),$$

with $\omega \approx 52°$. Suppose that the crack selects the depth d^* by the condition that $K_{II} = 0$, and we obtain that $d^* = 3.8h$. One can further confirm that the mode I path is stable in that, if a parallel crack at a depth d different from d^*, then K_{II} is nonzero and is in the direction that tends to deflect the crack back toward the depth d^*.

Mud crack in a thin film bonded to a substrate



Cracked SiN film of about 1 um thick, grown on silicon substrate. The contrast also indicates that the film was partially debonded. Courtesy of Dr. Qing Ma, of Intel Coorporation.

Curved crack paths can be simulated using numerical methods. For a given crack configuration, one solves the elasticity boundary value problem, and computes the stress intensity factors K_I and K_{II}. One then advances the crack by a small length in the direction, say, set by the criterion of maximum hoop stress. The path so selected should be essentially a mode I path. For a single crack tip, the precise length for each increment is unimportant, so long as it is much smaller than the representative size of the sample. To simulate simultaneous growth of multiple cracks, however, one has to know how much to advance each crack. An ingredient of time-dependence has to be introduced into the model. For example, if the solid is susceptible to subcritical cracking, the V-G relation provides the needed information. Once the energy release rate is calculated for every crack tip in a given configuration, one advances each crack according to the kinetic law for a small time step. Similarly, one can simulate the growth of a crack in three dimensions with a curved front by advancing each point on the crack front according to the kinetic law and its local energy release rate. Because the crack extends under the mode I conditions, the V-G curve can be obtained experimentally using a specimen containing a single straight mode I crack.

The regular finite element method meshes the geometry of the crack and uses a fine mesh near the crack tip. When the crack grows, remeshing is required. To circumvent these difficulties, an extended finite element method (XFEM) has been advanced. For the nodes around the crack tip, one adds enriching functions derived from the singular crack-tip stress field. For nodes on the crack faces, one adds the Heaviside function to represent the displacement jump. Consequently, the mesh can be coarse near the crack tip, and the elements need not to conform to the crack geometry. As the cracks grow, one updates the nodes to be enriched. No remeshing is necessary. The Figure below shows the simulated mud-crack pattern in tensile film bonded to a substrate (J. Liang, R. Huang, J.H. Prévost, and Z. Suo, Evolving crack patterns in thin films with the extended finite element method. *Int. J. Solids Structures* 40, 2343-2354 (2003) (http://www.deas.harvard.edu/suo/papers/137.pdf)