8. Nonlinear Fracture Mechanics I: J-integral and Crack-tip Field

References:
J. W. Hutchinson, *Notes on Nonlinear Fracture Mechanics* (http://imechanica.org/node/755);
Alan Zehnder, *Lecture Notes on Fracture Mechanics* (http://hdl.handle.net/1813/3075).

Deformation theory of plasticity. Under a monotonic loading, elastic-plastic deformation of a material may be described by a nonlinear stress-strain relationship within the context of small strains, which is essentially a nonlinear elasticity theory sometimes called *hypoelasticity*. Despite its limitations, the deformation theory provides useful insights into the elastic-plastic stress field near a crack tip and initiation of crack growth in ductile materials (low strength and high toughness) under monotonic loads.

The nonlinear stress-strain relationship can be developed by assuming a strain energy density function, $w(\varepsilon_{ij})$, such that

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}$$

Under uniaxial tension, a power-law stress-strain relationship is often adopted for the convenience of analysis:

$$\frac{\varepsilon}{\varepsilon_0} = \left(\frac{\sigma}{\sigma_0}\right)^n$$

where σ_0 may be considered as an effective tensile yield stress and ε_0 the associated tensile strain. The power exponent n is typically greater than 1. For many metals, n = 10~25. For the limiting cases, the material is perfectly plastic (no hardening) when $n = \infty$ and is perfectly elastic when $n = 1$.

Under multi-axial stresses, the power-law relation takes the general form:

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \frac{3}{2}\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\frac{s_{ij}}{\sigma_0}$$

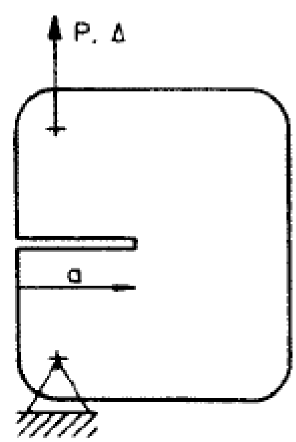
where $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}$ is the deviatoric stress tensor and $\sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}}$ is the effective shear stress. The material is assumed to be incompressible under the deformation theory of plasticity.

Energy release rate. Consider a cracked specimen. Under monotonic loading, the strain energy is

$$U(a,\Delta) = \int_0^\Delta P(u)du$$

where $P(u)$ is a nonlinear function depending on the crack length a. The potential energy is

$$\Pi(a,P) = U - P\Delta$$



Similar to LEFM, the energy release rate per unit thickness for straight-ahead crack growth can be defined as

$$G = -\left(\frac{\partial U}{\partial a}\right)_\Delta$$

Or equivalently, $G = -\left(\frac{\partial \Pi}{\partial a}\right)_P$. Two points are noted for the definition of energy release rate:

(i) In LEFM, $\Pi = -U$, and thus $G = \left(\frac{\partial U}{\partial a}\right)_P$. The same is not true in nonlinear fracture mechanics.

(ii) Strictly speaking, the definition of the energy release rate is only meaningful for linear or nonlinear elastic materials, for which the strain energy in the body can be fully released upon unloading. For elastic-plastic materials, part of the strain energy is unrecoverable due to plasticity. As the crack advances, only the elastic part of the strain energy is released. Nevertheless, a critical energy release rate may be determined as the toughness for the initiation of crack growth under monotonic loading.

J-integral. The contour integral previously discussed under the condition of small-scale yielding for LEFM can be extended to nonlinear fracture mechanics under the condition that there exists a strain energy density function in form of $w(\varepsilon_{ij})$. In a 2D specimen (plane stress or plane strain), the J-integral

$$J = \int_C \left(w n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) ds$$

is path-independent as long as the contour starts on one crack face and ends on the other. It was further shown that $J = G$ (Budiansky and Rice, J. Appl. Mech., pp. 201, 1973).

Crack-tip field (HRR solution). Consider a semi-infinite crack in an infinite body. Two sets of dimensional analyses lead to a formal solution for the stress field near the crack tip. First, given a single loading parameter *P*, the stress scales linearly with the applied load, i.e., $\sigma_{ij} \sim P$. By the power-law relationship, the strain scales with the load as $\varepsilon_{ij} \sim P^n$. Integration of the strain tensor with respect to the spatial coordinates gives the displacement with the same scaling, $u_i \sim P^n$. Consequently, the strain energy density, $w = \int \sigma_{ij} d\varepsilon_{ij} \sim P^{n+1}$, and the energy release rate or J-integral, $J \sim wa$. The dimensional consideration thus leads to a general form of the energy release rate (Ilyushin theorem):

$$J = \Omega a \sigma_0 \varepsilon_0 \left(\frac{\sigma_{appl}}{\sigma_0}\right)^{n+1}$$

where Ω is a dimensionless parameter that depends on the specimen geometry. This is analogous to the dimensional form of the energy release rate in LEFM: $G = \Omega \frac{\sigma^2 a}{E}$. Numerical values of Ω for a number of crack specimens are available in the ASTM handbook by C. F. Shih (1974).

Next, take the contour C of the J-integral to be a circle of radius r centered at the crack tip, and thus

$$J = \int_{-\pi}^{\pi} \left(w n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} \right) r d\theta$$

For J to be independent of the path radius r and to have a finite value as $r \to 0$, it is necessary that

$$w n_1 - \sigma_{ij} n_j \frac{\partial u_i}{\partial x_1} = \frac{f(\theta)}{r} \text{ as } r \to 0$$

Therefore, the strain energy density scales inversely with r, i.e., $w \sim \frac{1}{r}$. With the power-law relationship, the stress and strain must have the following scaling as $r \to 0$:

$$\varepsilon_{ij} \sim r^{-\frac{n}{n+1}},\ \sigma_{ij} \sim r^{-\frac{1}{n+1}}$$

Both the strain and stress are singular, but the singularity for $n > 1$ differs from the square-root singularity in LEFM for which $n = 1$. On the other hand, integration of the strain with respect to r gives the displacement with a scaling: $u_i \sim r^{\frac{1}{n+1}}$, which is non-singular.

Combining the two sets of dimensional analyses, we obtain the following:

$$\sigma_{ij} \sim P\, r^{-\frac{1}{n+1}},\ \varepsilon_{ij} \sim P^n r^{-\frac{n}{n+1}},\ u_i \sim P^n r^{\frac{1}{n+1}}$$

Since $J \sim P^{n+1}$ independent of r, we replace the loading parameter P in the above scaling with J as $P \sim J^{\frac{1}{n+1}}$, and obtain that

$$\sigma_{ij} \sim \left(\frac{J}{r}\right)^{\frac{1}{n+1}},\ \varepsilon_{ij} \sim \left(\frac{J}{r}\right)^{\frac{n}{n+1}},\ u_i \sim J^{\frac{n}{n+1}} r^{\frac{1}{n+1}}$$

The crack-tip field can be written as

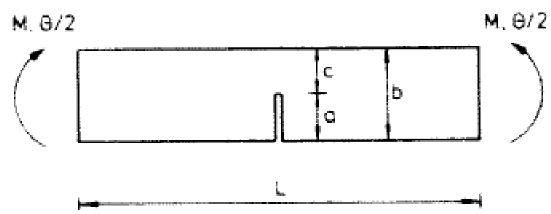
$$\sigma_{ij} = \sigma_0 \left(\frac{J}{\sigma_0 \varepsilon_0 r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(\theta, n)$$

$$\varepsilon_{ij} = \varepsilon_0 \left(\frac{J}{\sigma_0 \varepsilon_0 r} \right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}(\theta, n)$$

where the circumferential distributions are given by the dimensionless functions $\tilde{\sigma}_{ij}(\theta, n)$ and $\tilde{\varepsilon}_{ij}(\theta, n)$. The above crack-tip solution as well as numerical results for the circumferential functions were obtained independently by Hutchinson (*J. Mech. Phys. Solids* 16, pp. 13-31, 1968) and Rice and Rosengren (*J. Mech. Phys. Solids* 16, pp. 1-12, 1968), thus called *HRR solution*. It was noted that the dimensionless functions depend on whether plane stress or plane strain condition holds in the plastic zone. Similar to LEFM, the θ-dependence of the HRR field can be obtained separately for mode I and mode II cracks, and the analogous field for mode III can also be obtained analytically.

Remarks: (i) The HRR solution is a nonlinear extension of the K-field solution in LEFM. Here, *J* plays a similar role as *K* measuring the intensity of the singular stress field at the crack tip that depends on external loading and specimen geometry. The small scale yielding condition in LEFM is now replaced by a small-scale process zone condition. (ii) The dimensionless functions $\tilde{\sigma}_{ij}(\theta, n)$ and $\tilde{\varepsilon}_{ij}(\theta, n)$ are unique for finite *n* (strain hardening). For $n \to \infty$ (perfectly plasticity), however, they are not unique but depend on the full solution to the boundary value problem. This is a consequence of the fact that the field equations of perfectly plasticity are hyperbolic while those for finite *n* are elliptic. (iii) The HRR solution is valid only in an annular region around the crack tip, where the small strain deformation theory of plasticity is applicable. When this region is large compared to the fracture process zone and the region of large strains governed by finite strain plasticity, the crack tip field can be sufficiently characterized by a single parameter *J*. This restriction strongly depends on the crack configuration, strain hardening exponent, and plane strain or plane stress conditions.

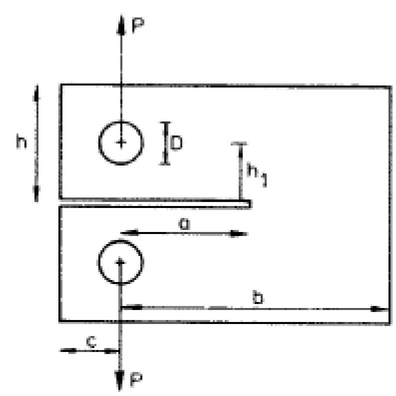
Deeply cracked test specimens. These specimens are designed such that a simple formula for *J* can be used to determine *J* as a function of load-displacement data directly from the measurements. For example, for the deeply cracked bend specimen as shown in the figure with *c*/*b* < 0.5, the J-integral was shown to be given by (Rice et al., 1973)



$$J = \frac{2}{c}\int_0^{\theta} M(\theta_{cr})\, d\theta_{cr}$$

where *M* is the applied bending moment per unit thickness of the specimen and $\theta_{cr} = \theta - \theta_0$: θ is the total angle of rotation at the load point, θ_0 is the reference rotation for an uncracked specimen (a = 0), and θ_{cr} is the contribution due to the presence of the crack. Therefore, the J-integral is simply 2/*c* times the area under the bending moment *M* vs θ_{cr} curve that can be measured experimentally.

A similar expression for J-integral applies to deeply cracked compact tension specimen:



$$J = \frac{2}{c}\int_0^{\Delta} P(\Delta_{cr})\, d\Delta_{cr}$$

Where *P* is the applied load per unit thickness of the specimen and $c = b - a$ is the length of the uncracked ligament. Often Δ_{cr} is replaced by the total displacement Δ since the reference displacement Δ_0 for an uncracked compact tension specimen is usually negligible.

The bending and compact tension specimens can be used to determine the critical *J* for initiation of crack growth as well as the resistance curve *J* vs Δa for small amount of crack growth (Shih et al., 1978).