

## Linear Elastic Fracture Mechanics

**Brittle fracture.** When a body of a material fractures by the propagation of a crack, the stress is intensified around the front of the crack, and is low elsewhere in the body. Except for a small zone around the front of the crack, the deformation of the body obeys the linear theory of elasticity. If this is the case, the body is said to undergo brittle fracture, under the small-scale yielding condition. The small zone, in which the linear theory of elasticity is invalid, is known as plastic zone, inelastic zone, or fracture process zone.

Thus, the three ideas are closely related: brittle fracture, small-scale yielding, and linear elastic fracture mechanics.

**The form of the singular field around the tip of a crack is universal, independent of the shape of the body and the distribution of the load.** The following idealizations are made:

- The crack consists of two flat planes.
- The front of the crack is a straight line. By modeling the crack front as a mathematical line, the linear elastic theory does not account for any process of fracture.
- The body obeys the linear elastic theory.
- The material is isotropic.
- The body deforms under the plane-strain conditions.
- The loading is mode I.

Within these idealizations, the field of stress around the front takes the form

$$\sigma_{ij}(r, \theta) = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta).$$

The dependence on  $r$  is square-root singular. The functions  $f_{ij}(\theta)$  are known functions. Consequently, the form of the field around the front is independent of the shape of the body and the distribution of the load.

**Relate the amplitude of the crack-tip field to the external boundary conditions.** The amplitude of the crack-tip field scales with a single parameter, the stress intensity factor  $K$ , which relates to the energy release rate as  $K = \sqrt{G\bar{E}}$ . The amplitude of the crack-tip field does depend on the shape of the body and the distribution of the applied load.

For a given body containing a crack, one *assumes* that the crack tip is a mathematical point, the crack a mathematical cut, and the body obeys the linear elasticity theory. The linearly elastic boundary-value problem determines  $K$ . Linearity and dimensional considerations dictate that the stress intensity factor should take the form

$$K = Y\sigma\sqrt{L},$$

where  $L$  is a representative length,  $\sigma$  is a stress characterizing the magnitude of the load, and  $Y$  is a dimensionless number depending on the shape of the body

and the distribution of the load. For example, for the Griffith crack, length  $2a$ , in a infinite body under a stress  $\sigma$ , the stress intensity factor is

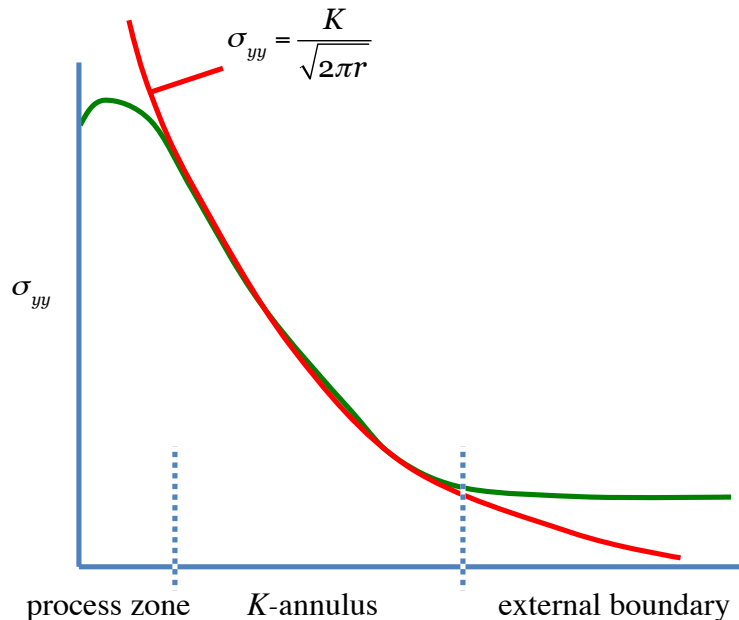
$$K = \sigma\sqrt{\pi a}.$$

**K-annulus.** A careful examination of how we use the square-root singular field clarifies why the Linear Elastic Fracture Mechanics works. The square-root singular field is invalid in the fracture process zone, where inelasticity prevails. The square-root singular field is also invalid on the size scale of the external boundary, where the external boundary conditions prevail.

Consider the Griffith crack again, assuming that the body is made of glass. Sketch the distribution of the stress  $\sigma_{yy}(x,0)$ . Discuss the field in three regions:

- Very close to the crack tip (less than 1 nm for the glass), atomic bonds are being stretched, and the linear elasticity theory is invalid. The stress is finite, below the theoretic strength of the glass. This region is known as the fracture process zone. The  $K$ -field is wrong in the fracture process zone. Denote the radius of the process zone by  $r_p$ .
- On the scale comparable and larger than the length of the crack  $a$ , the field is determined by the external boundary conditions. The  $K$ -field is wrong in this region also.
- In an annulus,  $r_p < r < a$ , the  $K$ -field is valid. We call this annulus the  $K$ -annulus, or  $K$ -dominant zone.

For a representative size of the crack,  $a = 1\mu\text{m}$ , the  $K$ -field is valid in the annulus  $1\text{nm} < r < 1\mu\text{m}$ , over several orders of magnitude in length.



**The Linear Elastic Fracture Mechanics.** Imagine two bodies of the same material, each containing a crack under an opening load. The two bodies may have different sizes and different shapes, and lengths of the cracks in the two bodies and the loads on the two bodies may also be different. Regardless of all these differences, the field in the  $K$ -annulus depends on the boundary conditions through the stress intensity factor. That is,  $K$  is the only messenger between the external boundary conditions and the fracture process.

If  $K$  is the same for the two bodies, the fracture processes at the tips of the two cracks must be identical. The doctrine of the linear elastic fracture mechanics has been so summarized:

***The same  $K$ , the same fracture process.***

**Fracture criterion phrased in terms of  $K$ .** Within the fracture mechanics, the simplest way to account for the fracture process is to stipulate that the crack extends when the stress intensity factor  $K$  reaches a critical value,  $K_c$ . This critical value  $K_c$  is known as the toughness.

The stress intensity factor is a *loading parameter*. The toughness is a *material parameter*. For a given material,  $K_c$  is determined by a fracture test. Representative values of toughness:

- Glass:  $K_c = 1\text{MPa}\sqrt{\text{m}}$ .
- Steel:  $K_c = 100\text{MPa}\sqrt{\text{m}}$ .
- Epoxy:  $K_c = 1\text{MPa}\sqrt{\text{m}}$ .

In applications, we use two handbooks. One contains the elasticity solutions of many cracked *geometries*, and the other contains the experimentally determined values of toughness for many *materials*. The crack does not grow if

$$K < K_c,$$

or

$$Y\sigma\sqrt{L} < K_c.$$

**The condition under which the  $K$ -annulus exists: small-scale yielding condition.** The  $K$ -annulus consists of two circles. When discussing The outer circle excludes any significant boundary effect, and the inner circle excludes the fracture process zone. Within the  $K$ -annulus, the field is well represented by the square-root singular field. This approach breaks down when the characteristic length scale of size of fracture process zone is comparable to the size of the body or size of the crack. This discussion identifies two size scales:

- (1) **The size of the fracture process zone**, in which the linear elastic theory is invalid. For glass, the linear elastic theory is valid all the way to a few atomic distances, so the fracture process zone encloses the bond-breaking atoms. For steel, the linear elastic theory is wrong in the region where steel deforms plastically, so that the fracture process zone encloses

the plastic zone around the crack tip. Representative size of the fracture process zone is nanometers for glass, and millimeters for steel.

(2) **The size representative of the external boundary conditions.**

For the Griffith crack, this size is the half length of the crack. For a channel crack in a brittle film bonded to a substrate, this size is the thickness of the film.

For the  $K$ -annulus to exist, the small-scale yielding condition must be satisfied:

**The fracture process zone « the external boundary zone**

**Size of plastic zone.** The Linear Elastic Fracture Mechanics (LEFM) requires the existence of the  $K$ -annulus, which in turn, requires the small scale yielding condition. We now estimate the fracture process zone size.

The elastic stress field scales as

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}}.$$

Let the yield strength by  $\sigma_Y$ . Approximately, the plastic zone size is given by

$$r_p \approx \frac{1}{2\pi} \left( \frac{K_c}{\sigma_Y} \right)^2.$$

The plastic zone size is expressed in terms of the material parameters. The plastic zone size itself is also a material parameter.

Glass:  $\sigma_Y \sim 10\text{GPa}$  and  $K_c \sim 1\text{MPa}\sqrt{\text{m}}$ .

$$r_p \approx \frac{1}{2\pi} \left( \frac{K_c}{\sigma_Y} \right)^2 = \frac{1}{2\pi} \left( \frac{10^6}{10^{10}} \right)^2 \sim 10^{-9}\text{m}.$$

High strength steel:  $\sigma_Y \sim 1.4\text{GPa}$  and  $K_c \sim 65\text{MPa}\sqrt{\text{m}}$ .

$$r_p \approx \frac{1}{2\pi} \left( \frac{K_c}{\sigma_Y} \right)^2 = \frac{1}{2\pi} \left( \frac{65 \times 10^6}{1.4 \times 10^9} \right)^2 \sim 0.3\text{mm}.$$

Tough steel:  $\sigma_Y \sim 0.35\text{GPa}$  and  $K_c \sim 180\text{MPa}\sqrt{\text{m}}$ .

$$r_p \approx \frac{1}{2\pi} \left( \frac{K_c}{\sigma_Y} \right)^2 = \frac{1}{2\pi} \left( \frac{180 \times 10^6}{0.35 \times 10^9} \right)^2 \sim 4\text{cm}.$$

The linear elastic fracture mechanics is difficult to apply for high-toughness, low-strength materials, because these materials have very large plastic zone size.

**Effect of specimen thickness. Plane strain fracture toughness.**

A thin sheet deforms under the plane-stress conditions, and the stress state is biaxial. A thick body deforms under the plane-strain conditions, and the stress state is triaxial. The difference is important for plastically deformable materials. For example, a thin sheet may fracture by necking, but a block may fracture by

void growth. The toughness is a function of the specimen thickness. The standard test requires that the fracture toughness be measured under the plane-strain conditions.

**The small-scale yielding condition in practice.** “Standard Test Method for Plane-Strain Fracture Toughness of Metallic Materials,” *ASTM Annual Book of Standards*, Part 10, American Society for Testing and Materials, Philadelphia, E23-72, pp. 273-289 (1980).

To measure the plane strain fracture toughness, the Standard requires that

$$a > 2.5 \left( \frac{K_c}{\sigma_Y} \right)^2$$

$$B > 2.5 \left( \frac{K_c}{\sigma_Y} \right)^2$$

$$W > 5.0 \left( \frac{K_c}{\sigma_Y} \right)^2$$

where  $a$  is the length of the crack,  $B$  the thickness of the specimen, and  $W$  the width of the specimen.

**Determine the energy release rate by solving a boundary-value problem.** Energy release rate represents the amplitude of the applied load, and the amplitude of the crack-tip field. For some problems, the energy release rate can be calculated without determining the detailed stress field. We have seen one example: the double-cantilever beam.

In general, one needs to solve the boundary-value problem. Once the boundary-value problem is solved, we can examine the near-tip field. For example, the stress a distance  $r$  ahead the tip is

$$\sigma_{\theta\theta} = \sqrt{\frac{G\bar{E}}{2\pi r}}.$$

The opening displacement a distance  $r$  behind the tip is

$$\delta = 8\sqrt{\frac{Gr}{2\pi\bar{E}}}.$$

A comparison between the above expression with the solution of the boundary-value problem gives the energy release rate.

Of course, we can also write the above two expressions in terms of the stress intensity factor  $K = \sqrt{G\bar{E}}$ . Thus, the stress a distance  $r$  ahead the tip is

$$\sigma_{\theta\theta} = \frac{K}{\sqrt{2\pi r}}.$$

The opening displacement a distance  $r$  behind the tip is

$$\delta = \frac{8K}{E} \sqrt{\frac{r}{2\pi}}.$$

**Griffith crack.** For example, consider the Griffith crack, a crack of length  $2a$  in an infinite plate, subject to remote stress  $\sigma$ . The boundary-value problem was solved by Inglis (1913). The field in the body is expressed in analytical terms. For example, the stress ahead of the crack is given by

$$\sigma_{yy} = \frac{\sigma x}{\sqrt{x^2 - a^2}}, |x| > a$$

The distance of a point  $x$  ahead of the crack tip is given by  $r = x - a$ . Replace  $x$  by  $r$ , and we obtain that

$$\sigma_{yy} = \frac{\sigma(r+a)}{\sqrt{(r+2a)r}}$$

When the tip of the crack is approached,  $r \ll a$ , we have

$$\sigma_{yy} = \sigma \sqrt{\frac{a}{2r}}.$$

This crack-tip field is obtained from the boundary-value problem, and it recovers the form determined by the eigenvalue problem, namely,

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}},$$

A comparison of the two expressions gives

$$K = \sigma \sqrt{\pi a}.$$

This is the stress intensity factor for the Griffith crack.

Alternatively, one can determine the stress intensity factor by using the displacement field. The boundary-value problem solved by Inglis also gives the opening displacement of the crack:

$$\delta(x) = \frac{4\sigma}{E} \sqrt{a^2 - x^2}, |x| < a.$$

The distance of a point  $x$  behind the crack tip is given by  $r = a - x$ . Replace  $x$  by  $r$ , and we obtain that

$$\delta(x) = \frac{4\sigma}{E} \sqrt{(r+2a)r}.$$

When the tip of the crack is approached,  $r \ll a$ , we have

$$\delta(x) = \frac{4\sigma}{E} \sqrt{2ar}.$$

This crack-tip field is obtained from the boundary-value problem, and it recovers the form determined by the eigenvalue problem, namely,

$$\delta = \frac{8K}{E} \sqrt{\frac{r}{2\pi}}.$$

A comparison of the two expressions gives

$$K = \sigma\sqrt{\pi a}.$$

**Handbooks for  $K$ .** Stress intensity factor for a given cracked body is determined by solving a boundary-value problem. Many configurations of cracked bodies have been solved. The results are collected in handbooks (e.g., H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook*, Del Research, St. Louis, MO., 1995). In general, the stress intensity factor takes the form

$$K = Y\sigma\sqrt{a},$$

where  $\sigma$  is an applied stress,  $a$  is a length scale characterize the crack geometry, and  $Y$  is a dimensionless number. Two examples follow.

For a crack at the edge of a semi-infinite plane, the stress intensity factor is

$$K = 1.1215\sigma\sqrt{\pi a}.$$

For a penny-shaped crack in an infinite body, the stress intensity factor is

$$K = \frac{2}{\pi}\sigma\sqrt{\pi a}.$$

For the two examples, can you explain why the first example has a larger stress intensity factor than the Griffith crack, and the second example has a smaller one? In any event, the difference in the stress intensity factors for the three cases is small. Thus, for a small crack in a large body, the stress intensity factor is determined by the size of the crack, but is insensitive to the shape of the crack.

**Linear superposition.** Elasticity problem is linear. For a given body with a given crack, if a force  $P$  causes the stress intensity factor  $K = \alpha P$ , and force  $Q$  causes the stress intensity factor  $K = \beta Q$ . The combined action of the forces  $P$  and  $Q$  causes the stress intensity factor  $K = \alpha P + \beta Q$ .

**Finite element method to determine  $K$ .** For a complicated structure with a crack, once can determine the elastic field using the finite element method, and then extract from the field the stress intensity factor. A brute force method is that you use the finite element method to determine the displacement field, and then fit the crack opening to

$$\delta = \frac{8K}{E} \sqrt{\frac{r}{2\pi}},$$

with  $K$  as the fitting parameter. There are a number of more clever methods. We'll mention them later at suitable points.

**Rank materials according to toughness.** Ranking materials according to toughness is similar to ranking universities according to excellence. The rankings depend on how we quantify toughness of materials and excellence of universities. Irwin's formula relates the two loading parameters  $K$  and  $G$  as

$$G = \frac{K^2}{E}.$$

Consequently, the two material properties, the critical energy release rate  $G_c$  and the critical stress intensity factor  $K_c$  are similarly related:

$$G_c = \frac{K_c^2}{E}.$$

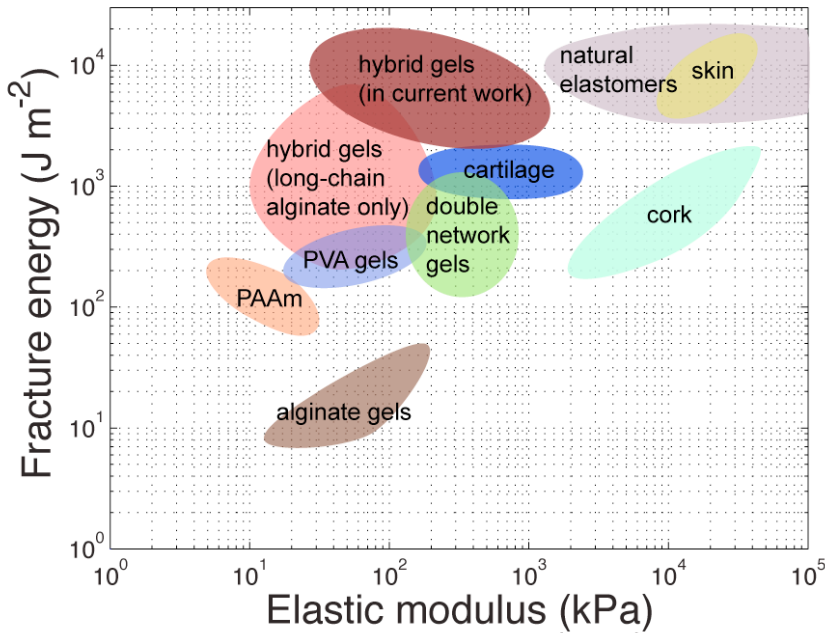
The critical energy release rate  $G_c$  is also known as the fracture energy  $\Gamma$ , or tear energy. Here are some representative values

Material	Modulus $E$ , GPa	$G_c$ , J/m <sup>2</sup>	$K_c$ , MPa m <sup>1/2</sup>
Steel 4340	210	25,000	50
Aluminum	70	8,000	24
Natural rubber	10 <sup>-3</sup>	10,000	0.1
Polystyrene	3	33	1
Silica	70	4	0.5

It is harmless to call either  $G_c$  or  $K_c$  the toughness of a material, so long as the context makes it clear whether we mean  $G_c$  or  $K_c$ . When we rank materials according to toughness, however, the order of the ranking depends on whether we use  $G_c$  or  $K_c$ . For example, silica is tougher than rubber according to their values of critical stress intensity factor, but rubber is tougher than silica according to their values of fracture energy.

**Ashby diagram.** Instead of ranking materials using a single property, one can always list many properties for every material. To compare a large number of materials graphically, one can choose two properties and construct an Ashby diagram. Here is an example plotting the fracture energy and Young's modulus of soft materials. J.Y. Li, W.R.K. Illeperuma, Z. Suo, J.J. Vlassak, Hybrid hydrogels with extremely high stiffness and toughness. Submitted for publication.





**Fracture of brittle adhesive.** Epoxy has an elastic modulus on the order of 1 GPa and a critical energy release rate on the order of 1 MPam<sup>1/2</sup>. Epoxy is often used as an adhesive to bond metals like aluminum. Both values are much smaller than those of aluminum.

For a large plate of epoxy containing a crack of length  $2a$ , subject to stress  $\sigma$ , the energy release rate is

$$G = \pi \frac{\sigma^2 a}{E_{\text{epoxy}}}.$$

Next consider a thin layer of epoxy used as an adhesive to bond two blocks of aluminum. Consider a crack in the epoxy parallel to the interfaces. The length of crack,  $a$  is assumed to be much larger than the thickness of the adhesive. The two blocks of the aluminum is pulled by a stress  $\sigma$ . In this case, the energy release rate is

$$G = \pi \frac{\sigma^2 a}{E_{\text{Al}}}.$$

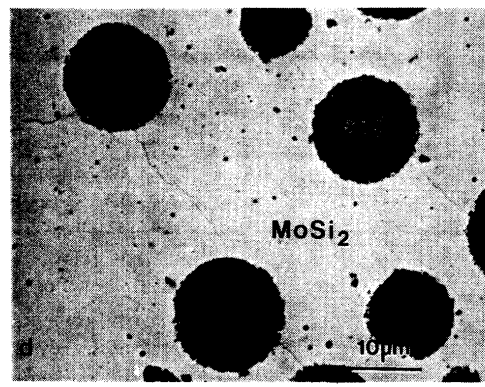
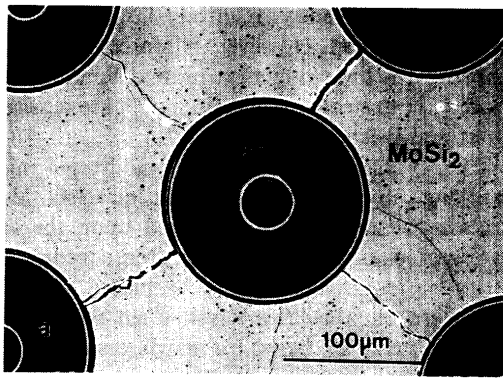
In reaching this result, we have regarded the thin layer of adhesive as a small-scale feature and neglected its effect on the energy release rate. The relevant modulus is that of aluminum.

In both cases, the critical energy release rate is that of the epoxy. We assume that cracks in the bulk epoxy and thin layer of epoxy are comparable in length. Because of the large difference in the elastic moduli of the two materials,

the stress to fracture the bulk epoxy is much lower than the stress to fracture a thin layer of epoxy between two blocks aluminum.

- N.A. Fleck, J.W. Hutchinson and Z. Suo, "Crack path selection in a brittle adhesive layer," *Int. J. Solids and Structures*, 27, 1683-1703 (1991). <http://www.seas.harvard.edu/suo/papers/009.pdf>

**Avoid cracking in the matrix around an inclusion.** T.C. Lu, J. Yang, Z. Suo, A.G. Evans, R. Hecht, and R. Mehrabian, Matrix cracking in intermetallic composites caused by thermal expansion mismatch. *Acta Metall. Mater.* 39, 1883-1890 (1991). You may wish to take a look at analytical details and micrographs of cracks around inclusions (<http://www.seas.harvard.edu/suo/papers/013.pdf>).



A fiber of radius  $R$  is embedded in an infinite matrix. For simplicity, we assume that the fiber and the matrix are both elastic, having similar Young's modulus  $E$  and Poisson's ratio  $\nu$ . The two materials are bonded at an elevated temperature. When they are cooled to the room temperature, the two materials develop stresses due to the mismatch in thermal expansion. Set the representative stress by

$$\sigma = E\Delta\alpha\Delta T / (1-\nu),$$

where  $\Delta\alpha$  is the difference in the coefficients of thermal expansion of the two materials, and  $\Delta T$  the change in the temperature. This stress, as shown by the elasticity solution, corresponds to the magnitude of the longitudinal stress or twice that of the transverse stress in the cylinder.

When the cylinder is under compression, the matrix is under a tensile hoop stress, which decays far away from the interface. To evaluate if this stress field will cause the matrix to fracture, we place a crack in the matrix, and ask if this crack will grow. It is sensible to place the crack near the interface and in the radial direction, where the stress is tensile and has a large magnitude.

The choice of the length of the crack requires some thought. When the length of the crack approaches zero, the stress intensity factor vanishes. When the length of the crack approaches infinity, the stress intensity factor also

vanishes. Sketch the trend of the stress intensity factor as a function of the crack length. This elastic boundary value problem has been solved analytically, giving the stress intensity factor:

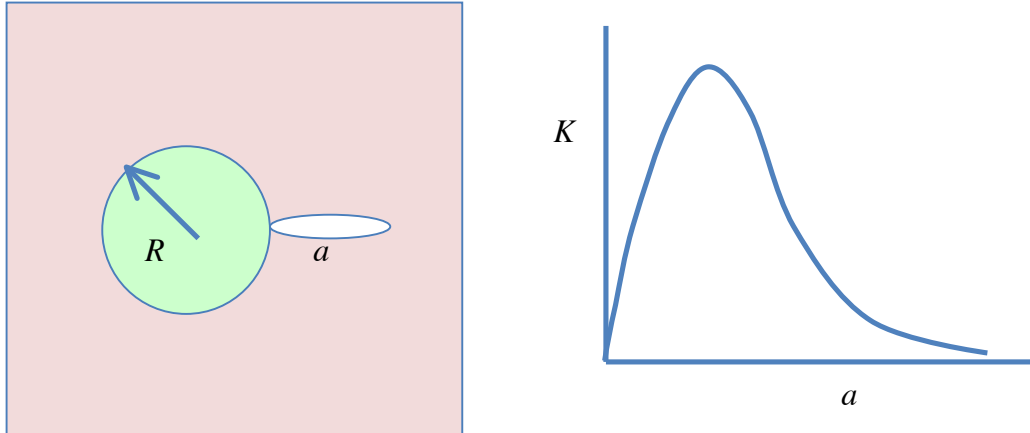
$$\frac{K}{\sigma\sqrt{R}} = \left(\frac{\pi}{8}\right)^{1/2} \left(\frac{a}{R}\right)^{1/2} \left(1 + \frac{a}{R}\right)^{-3/2}.$$

If we can estimate the flaw size due to a certain fabrication process, we can set the crack size to this flaw size, and then calculate the stress intensity factor.

Even if we know nothing about the flaw, we may proceed as follows. The stress intensity factor attains a maximum,  $K_{\max} = 0.24\sigma\sqrt{R}$ , at  $a/R = 0.5$ . No matrix flaw can grow provided this maximum stress intensity factor is below the fracture toughness, namely,

$$0.24 \frac{E\Delta\alpha\Delta T}{1-\nu} \sqrt{R} < K_c.$$

This condition invokes well defined quantities, and does not rely on any knowledge of the flaw. In particular, this condition predicts that a critical radius of the fiber exists, below which no flaw in the matrix can grow. Inserting representative values,  $E = 10^{11} \text{ Pa}$ ,  $\nu = 0.3$ ,  $\Delta\alpha = 10^{-5} \text{ K}^{-1}$ ,  $\Delta T = 1000 \text{ K}$ ,  $K_c = 10^6 \text{ Pa}\sqrt{\text{m}}$ , we estimate that  $R_c \approx 10 \mu\text{m}$ .



**Boundary-value problems of several types.** A crack advancing in a body under a load poses boundary-value problems of several types.

- Type A. A boundary-value problem that includes the external boundary conditions, but assumes the body to be linearly elastic all the way to the crack tip. Boundary-value problems of this type have been solved since Inglis (1913). The solutions of this type constitute the bulk of the Linear Elastic Fracture Mechanics. See the Tada handbook.

- Type B. A boundary-value problem that excludes both the external boundary conditions and the inelastic fracture process. Strictly speaking, this type has been solved only once, by Williams (1957). Now we understand that this problem only needs to be solved once, because the form of the resulting field, the square-root singular field, is universal, independent of the external boundary conditions and the inelastic fracture process. The amplitude of the square-root singular field is the stress intensity factor, which cannot be determined without prescribing the external boundary conditions.
- Type C. A boundary-value problem that includes both the external boundary conditions and the inelastic fracture process. Remarkably, boundary-value problem of this most intuitive type have been solved rarely. The first useful analysis of Type C was due to Dugdale (1960). We will study the Dugdale model with care later in the course.

**Applications of the Linear Elastic Fracture Mechanics.** When the size of the fracture process zone is small compared to the size of the specimen (i.e., the size of the crack, the size of the ligament, the thickness of the specimen, etc.), the specimen is said to fracture under the small-scale yielding condition.

Under the small-scale yielding condition, the square-root singular field prevails in an annulus. The inner circle of the annulus is larger than the size of the fracture process zone, and the outer circle of the annulus is smaller than the size of the specimen. Consequently, the stress intensity factor is the only messenger between the external boundary conditions and the fracture process zone.

The stress intensity factor  $K$  serves as a loading parameter for a cracked body. Alternatively, the energy release rate  $G$  serves as a loading parameter for a cracked body. The two loading parameters are equivalent, and are connected through Irwin's formula:

$$G = \frac{K^2}{E}.$$

The Linear Elastic Fracture Mechanics can be phrased in two equivalent forms, using either  $K$  or  $G$ . I'll mostly use  $G$ , but you can paraphrase in terms of  $K$ .

We have described quite a few applications of the linear elastic fracture mechanics in a previous lecture (<http://imechanica.org/node/7531>). Subsequent lectures will describe more applications:

- Resistance curve (<http://imechanica.org/node/7674>)
- Fatigue (<http://imechanica.org/node/7705>)
- Stress corrosion (<http://imechanica.org/node/7729>)