On the space of effective infinitesimal isometries for simply connected surfaces

Hussein Nassar*

Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA

We report on a recent theorem regarding the dimension and symplectic structure of the space of effective infinitesimal isometries for simply connected surfaces [1]. For the most straightforward setup, consider a simply connected surface $(x, y, z(x, y))$ where z is periodic. A pair of symmetric two-dimensional second-order tensors $(\mathbf{E}, \boldsymbol{\chi}) \in S^2 \times S^2$ is an *effective (infinitesimal) isometry* if there exists a periodic displacement $(\tilde{\mathbf{u}}, \tilde{w})$ such that

$$\begin{bmatrix} u_\alpha \\ w \end{bmatrix} \equiv \begin{bmatrix} E_{\alpha\beta}x_\beta + z\chi_{\alpha\beta}x_\beta \\ -\frac{1}{2}\chi_{\mu\nu}x_\mu x_\nu \end{bmatrix} + \begin{bmatrix} \tilde{u}_\alpha \\ \tilde{w} \end{bmatrix} \tag{1}$$

has a vanishing membrane strain $\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}^s\mathbf{u} + \boldsymbol{\nabla}w \otimes^s \boldsymbol{\nabla}z = \mathbf{0}$. Call $\mathcal{L} \subset S^2 \times S^2$ the space of effective infinitesimal isometries. Then, the theorem states that $\mathcal{L}$ is a *Lagrangian subspace* of $S^2 \times S^2$ equipped with the symplectic form

$$\omega((\mathbf{E}_1, \boldsymbol{\chi}_1), (\mathbf{E}_2, \boldsymbol{\chi}_2)) = \mathbf{E}_1 : \operatorname{cof} \boldsymbol{\chi}_2 - \mathbf{E}_2 : \operatorname{cof} \boldsymbol{\chi}_1. \tag{2}$$

That is dim $\mathcal{L} = 3$ and $\omega_{|\mathcal{L}} \equiv 0$. Practically, the theorem implies that a thin periodic shell with a simply connected mid-surface is able to efficiently resist (i.e., by membrane action) exactly three independent loading modes out of the possible six tension, shear, bending and twisting modes, and will be compliant (i.e., deform by pure flexure) under exactly three independent modes. Thus, corrugations and wrinkles, smooth or not, cannot stiffen deformation modes without softening others, except by altering the topology. The theorem also allows to establish some microstructure-independent relationships among the effective Poisson coefficients and the other effective Kirchhoff-Love moduli. The proof relies on two key ingredients: the static-geometric analogy, and the Hill-Mandel lemma. It is in part reminiscent of techniques employed in graphic statics and in the theory of composites (e.g., the CLM theorem [2]).

References

[1] H. Nassar (2026) “A topological counting rule for shells”, in *Communications Physics*, in press.

[2] A. V. Cherkaev, K. A. Lurie, and G. W. Milton (1992). “Invariant properties of the stress in plane elasticity and equivalence classes of composites”, *Proceedings of the Royal Society A*, **438**, 519.

*Corresponding author: hnassar@uh.edu