Acta Mech Sin (2009) 25:265–270
DOI 10.1007/s10409-008-0208-4

RESEARCH PAPER

New exact solutions for free vibrations of rectangular thin plates by symplectic dual method

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Received: 4 May 2008 / Revised: 11 August 2008 / Accepted: 3 September 2008 / Published online: 11 November 2008

Abstract The separation of variables is employed to solve Hamiltonian dual form of eigenvalue problem for transverse free vibrations of thin plates, and formulation of the natural mode in closed form is performed. The closed-form natural mode satisfies the governing equation of the eigenvalue problem of thin plate exactly and is applicable for any types of boundary conditions. With all combinations of simply-supported (S) and clamped (C) boundary conditions applied to the natural mode, the mode shapes are obtained uniquely and two eigenvalue equations are derived with respect to two spatial coordinates, with the aid of which the normal modes and frequencies are solved exactly. It was believed that the exact eigensolutions for cases SSCC, SCCC and CCCC were unable to be obtained, however, they are successfully found in this paper. Comparisons between the present results and the FEM results validate the present exact solutions, which can thus be taken as the benchmark for verifying different approximate approaches.

Keywords Classical theory of thin plate · Frequency · Free vibrations · Symplectic dual method · Exact solution

1 Introduction

The exact solutions for the free vibration problem of a rectangular thin plate can be obtained only for the case with at least two opposite edges simply supported by the semi-inverse method. How to obtain exact solutions for other cases is a bottleneck for further development of solution methods in elastic mechanics.

For the transverse free vibrations of plates, the natural mode $W(x, y)$ is related to two spatial coordinates x and y, and there exists a corresponding eigenvalue for each of them. In semi-inverse method, an eigenvalue and corresponding eigenfunctions are postulated to satisfy the boundary conditions of two opposite edges simply supported, and the problems with double eigenvalues are transformed to the problems with single eigenvalue. Except the case with at least two opposite edges simply supported, it is impossible to postulate such an eigenvalue and eigenfunctions that the boundary conditions along any two opposite edges can be satisfied, which is evidently a strong limitation for applying the inverse method even though it is a dominant method for 2D elastic problems.

The thin plate vibration problems have been studied extensively, including the pioneers [1,2] and Timoshenko's work [3] from energy point of view. Graff [4] summarized the exact normal modes and frequency equations of rectangular thin plates with at least two simply supported opposite edges. In the determination of exact solutions for free vibration of plates, derivation of the general mathematic expression of natural mode satisfying the governing equation of the eigenvalue problem is the most important and difficult problem, and thus many efforts were devoted to the development of approximate solutions with a high level of precision [5], and most approximate methods were established on the basis of series expansion. Due to its high versatility and conceptual simplicity, Ritz method is one of the most important approximate methods, in which the displacement boundary conditions are satisfied exactly but the governing equations and the natural boundary conditions are satisfied approximately, e.g. it was used by Leissa [6] to study the transverse free

This project was supported by the National Natural Science Foundation of China (10772014).

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vibrations of plates. Literature reviews can be found in Refs. [7–9], and some new representative approximate results for the transverse free vibration of thin plate are briefly reviewed below.

Gorman [10,11] obtained a series of solutions by the superposition method, in which the natural modes were expressed in trigonometric and hypermetric series, and the number of terms in series depended on the requirements of precision. In the novel superposition method introduced by Kshirsagar and Bhaskar [12], the infinite series counterparts of conventional Levy-type closed-form expressions were used to simplify the solution procedure without any compromise on accuracy. Filipich and Rosales [13] developed a variational method for the frequency analysis of a free rectangular thin plate within the Germain–Lagrange theory, and Seok et al. [14] studied the free vibration of a cantilevered plate based on a variational approximation approach, in which the differential equation and the conditions on either side of the width were satisfied exactly, and the conditions at the free and fixed edges were satisfied variationally. Hedrih [15] investigated the free and forced transversal vibrations of an elastically connected double-plate system based on the classical theory of thin plate and mode superposition method. Ouyang and Zhong [16], Zhong and Zhang [17], Bao and Deng [18] applied Hamilton dual method to the analyses of modes and frequencies of thin plates, in which the natural modes in their solutions were expressed in the forms of symplectic eigenfunction expansions rather than in the closed form, and Cen et al. gave some Hamiltonian dual differential equations of thin plates in Ref. [19].

Some new exact solutions for the transverse free vibrations of rectangular thin plate are obtained in the present paper, which is organized as follows. The Hamiltonian form of the eigenvalue problem of thin plate is reviewed in Sect. 2, and solved by using the separation of variables in Sect. 3. The exact solutions for cases SSCC, SCCC and CCCC, which are not available in literature, are obtained in Sect. 4, followed by numerical experiments and comparisons with FEM results. Finally, some conclusions are drawn in Sect. 6.

2 Hamiltonian form of eigenvalue problem

Let us consider a perfectly elastic, homogeneous, isotropic thin plate with uniform thickness h, Young's modulus E, Poisson's ration υ and mass volume density ρ. It is assumed that the thickness h is small compared with in-plane dimensions, as shown in Fig. 1. We take the xy plane for the middle plane of the plate and assume the deflection $w(x, y, t)$ to be small compared with the thickness h. The partial differential equation for free vibrations of thin plates can be written in terms of deflection as

$$\nabla^2\nabla^2 w(x, y, t) + \frac{\rho h}{D}\frac{\partial^2 w(x, y, t)}{\partial t^2} = 0, \tag{1}$$

where $D = Eh^3/12(1-\upsilon^2)$ is the bending rigidity, and ∇^2 is the Laplacian. It is well-known that the normal vibrations of an elastic linear system are harmonic, therefore the deflection in normal vibrations of thin plate can be assumed to be

$$w(x, y, t) = W(x, y)e^{i\omega t}, \tag{2}$$

where ω is the natural frequency, and $i^2 = -1$. Substitution of Eq. (2) into Eq. (1) yields a four order partial differential equation involving natural mode $W(x, y)$

$$\nabla^2\nabla^2 W(x, y) = k^4 W(x, y), \quad k^4 = \rho h\omega^2/D, \tag{3}$$

which is the governing equation of the eigenvalue problem of thin plates. The Hamiltonian dual form of Eq. (3) can be obtained as [14]

$$\boldsymbol{v}' = \boldsymbol{H}\boldsymbol{v}, \tag{4}$$

where $\boldsymbol{H}$ is a Hamiltonian differential operator matrix, $\boldsymbol{v}' = \partial\boldsymbol{v}/\partial x$, here $\boldsymbol{v}$ is the state vector composed of dual variables. And they have the forms

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\upsilon\partial^2/\partial y^2 & 0 & 0 & -1/D \\ -D(1-\upsilon^2)\partial^4/\partial y^4 + \omega^2\rho h & 0 & 0 & \upsilon\partial^2/\partial y^2 \\ 0 & 2D(1-\upsilon)\partial^2/\partial y^2 & -1 & 0 \end{bmatrix}, \tag{5}$$

$$\begin{aligned} \boldsymbol{v}^{\mathrm{T}} &= [\, W \quad \Theta \quad Q \quad M \,] \\ &= [\, W \quad \partial W/\partial x \quad -Q_x \quad M_x \,], \end{aligned} \tag{6}$$

where the bending moment and shear force are given by

$$\begin{aligned} M_x &= -D\left(\frac{\partial^2 W}{\partial x^2} + \upsilon\frac{\partial^2 W}{\partial y^2}\right), \\ Q_x &= -D\left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\upsilon\frac{\partial^3 W}{\partial x\partial y^2}\right]. \end{aligned} \tag{7}$$

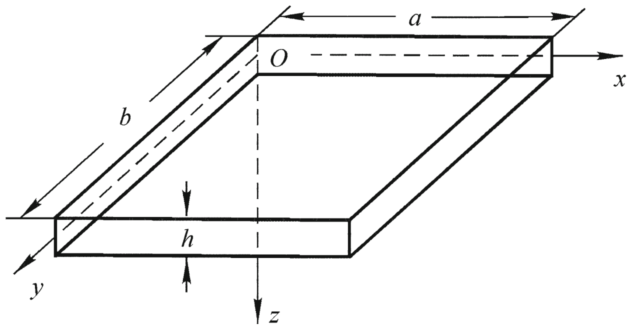

Fig. 1 Thin plate with coordinate convention

3 The separation of spatial variables

It is superior to solve Eq. (4) instead of Eq. (3), since the order of the differential equation is reduced from 4 [see Eq. (3)] to 1 [see Eq. (4)] and the separation of variables can be applied to the spatial variables. The separation-of-variable solution of Eq. (4) can be written as

$$\boldsymbol{v} = \boldsymbol{Y}(y)X(x), \tag{8}$$

where

$$\boldsymbol{Y}(y)^{\mathrm{T}} = [\bar{W}(y), \bar{\Theta}(y), \bar{Q}(y), \bar{M}(y)]. \tag{9}$$

Substituting Eq. (8) into Eq. (4) results in

$$\frac{\partial X}{\partial x} = \mu X, \tag{10a}$$
$$\boldsymbol{HY}(y) = \mu \boldsymbol{Y}(y), \tag{10b}$$

where μ is the eigenvalue of Hamiltonian matrix $\boldsymbol{H}$, and corresponds to spatial coordinate x. Integrating Eq. (10a) and substituting the results into Eq. (8) lead to

$$\boldsymbol{v} = (a_1 \mathrm{e}^{\mu x} + b_1 \mathrm{e}^{-\mu x})\boldsymbol{Y}(y). \tag{11}$$

A property of the eigenvalues of Hamilton matrix $\boldsymbol{H}$ is used in Eq. (11), i.e., if μ is an eigenvalue of $\boldsymbol{H}$, then $-\mu$ is also an eigenvalue of $\boldsymbol{H}$, and thus Eq. (10b) can be written as $\boldsymbol{HY}(y) = \pm\mu\boldsymbol{Y}(y)$, where $\boldsymbol{Y}(y)$ is a symplectic eigenfunction vector satisfying the symplectic conjugate orthogonal relations. It can be found from Eq. (11) that

$$W(x, y) = (a_1 \mathrm{e}^{\mu x} + b_1 \mathrm{e}^{-\mu x})\bar{W}(y), \tag{12}$$

where the symplectic eigenfunction $\bar{W}(y)$ can be solved from Eq. (10b). Eliminating $\bar{\Theta}$, $\bar{Q}$ and $\bar{M}$ from Eqs. (10b) and (9), we obtain a four order differential equation involving the symplectic eigenfunction $\bar{W}(y)$:

$$\frac{\partial^4 \bar{W}}{\partial y^4} + 2\mu^2 \frac{\partial^2 \bar{W}}{\partial y^2} + (\mu^4 - k^4)\bar{W} = 0. \tag{13}$$

For the above homogeneous equation (13), its particular solution has the form $\bar{W}(y) = \mathrm{e}^{\lambda y}$, and λ is the eigenvalue with respect to coordinate y. Substituting the particular solution into Eq. (13) results in the following characteristic equation

$$(\lambda^2 + \mu^2)^2 = k^4. \tag{14}$$

It is noteworthy that the two eigenvalues λ and μ and the natural frequency ω in Eq. (14) are all unknowns. It can be found from Eq. (14) that the Eigenvalues λ and μ can only be real or pure imaginary numbers for the free vibration of thin plates, and different $(\lambda^2 + \mu^2)$ will correspond to different natural frequency ω. Solving Eq. (14) gives us characteristic roots

$$\lambda_{1,2} = \pm \mathrm{i}\alpha_1, \quad \lambda_{3,4} = \pm\alpha_2, \tag{15a}$$

and

$$\mu_{1,2} = \pm \mathrm{i}\beta_1, \quad \mu_{3,4} = \pm\beta_2, \tag{15b}$$

where

$$\alpha_1 = \sqrt{k^2 + \mu^2}, \quad \alpha_2 = \sqrt{k^2 - \mu^2}, \tag{16a}$$
$$\beta_1 = \sqrt{k^2 + \lambda^2}, \quad \beta_2 = \sqrt{k^2 - \lambda^2}. \tag{16b}$$

Then the general solution of Eq. (13) can be written as

$$\bar{W}(y) = A_1 \cos\alpha_1 y + B_1 \sin\alpha_1 y + C_1 \cosh\alpha_2 y + D_1 \sinh\alpha_2 y, \tag{17}$$

and the natural mode in Eq. (12) becomes

$$W(x, y) = (a_1 \mathrm{e}^{\mathrm{i}\beta_1 x} + b_1 \mathrm{e}^{-\mathrm{i}\beta_1 x} + c_1 \mathrm{e}^{\mathrm{i}\beta_2 x} + d_1 \mathrm{e}^{-\mathrm{i}\beta_2 x})\bar{W}(y). \tag{18}$$

By substituting Eq. (18) into Eq. (3), we can verify that $W(x, y)$ is an exact solution of Eq. (3), i.e. natural mode $W(x, y)$ given in Eq. (18) satisfies Eq. (3) exactly. For the convenience of solution, Eq. (18) is rewritten as

$$W(x, y) = \Phi(x)\bar{W}(y), \tag{19}$$

where

$$\begin{aligned}\Phi(x) &= a_1 \mathrm{e}^{\mathrm{i}\beta_1 x} + b_1 \mathrm{e}^{-\mathrm{i}\beta_1 x} + c_1 \mathrm{e}^{\mathrm{i}\beta_2 x} + d_1 \mathrm{e}^{-\mathrm{i}\beta_2 x} \\ &= A_2 \cos\beta_1 x + B_2 \sin\beta_1 x \\ &\quad + C_2 \cosh\beta_2 x + D_2 \sinh\beta_2 x.\end{aligned} \tag{20}$$

It should be pointed out that the two functions $\Phi(x)$ and $\bar{W}(y)$ can not be solved independently except for the case with at least two opposite edges simply supported.

4 Normal mode and frequency equation

As the natural mode $W(x, y)$ has been obtained in the form of Eq. (19), the remaining problems are to determine the eight unknown constants in Eq. (19) and the frequency equation by means of the eight boundary conditions of the rectangular thin plate.

Table 1 The eigensolutions for the cases S-S, S-C and C-C

	Eigenvalue equations	Normal symplectic eigenfunctions
S-S	$\sin\alpha_1 b = 0$	$\bar{W}(y) = \sin\alpha_1 y$
S-C	$\alpha_2\tan\alpha_1 b = \alpha_1\tanh\alpha_2 b$	$\bar{W}(y) = \sin\alpha_1 y - \frac{\sin\alpha_1 b}{\sinh\alpha_2 b}\sinh\alpha_2 y$
C-C	$\frac{1-\cos\alpha_1 b\cosh\alpha_2 b}{\sin\alpha_1 b\sinh\alpha_2 b} = \frac{\alpha_1^2-\alpha_2^2}{2\alpha_1\alpha_2}$	$\bar{W}(y) = -\cos\alpha_1 x + \frac{\alpha_2}{\alpha_1}k_1\sin\alpha_1 y + \cosh\alpha_2 y - k_1\sinh\alpha_2 y$

This paper considers the simply supported and clamped boundary conditions only. All together there are six combinations for these two types of boundary conditions, i.e., SSSS, SCSS, SCSC, SSCC, SCCC and CCCC, among which only the first three cases with at least two opposite simply-supported edges can be solved by the inverse method. However, all the six cases can now be solved by the present method, in which the formulations of $\Phi(x)$ and $\bar{W}(y)$ can be performed independently with the same procedure, as shown in the following solution steps, but the quantities $\alpha_1, \alpha_1, \beta_1, \beta_1$, and ω in these two functions must be solved simultaneously. As an example, let us solve the case S-C, wherein the edge y=0 is simply supported (S) and the edge $y = b$ is clamped (C), to show the solution steps for $\bar{W}(y)$. The boundary conditions for the edge $y = 0$ are

$$W(x,0) = 0 \Rightarrow \bar{W}(0) = 0, \quad \frac{\partial^2 W(x,0)}{\partial y^2} + \upsilon\frac{\partial^2 W(x,0)}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 \bar{W}(0)}{\partial y^2} = 0. \quad (S) \tag{21}$$

The boundary conditions for the edge $y = b$ are

$$W(x,b) = 0 \Rightarrow \bar{W}(b) = 0, \quad \frac{\partial W(x,b)}{\partial y} = 0 \Rightarrow \frac{\partial \bar{W}(b)}{\partial y} = 0. \tag{22}$$

Introducing the solution (17) of Eq. (13) into Eqs. (21) and (22) leads to a homogeneous algebraic system of four equations for unknown constants A_1, B_1, C_1 and D_1

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ -\alpha_1^2 & 0 & \alpha_2^2 & 0 \\ \cos\alpha_1 b & \sin\alpha_1 b & \cosh\alpha_2 b & \sinh\alpha_2 b \\ -\alpha_1\sin\alpha_1 b & \alpha_1\cos\alpha_1 b & \alpha_2\sinh\alpha_2 b & \alpha_2\cosh\alpha_2 b \end{pmatrix} \times \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{23}$$

From Eq. (23), one can obtain

$$A_1 = C_1 = 0, \tag{24}$$

then Eq. (23) is reduced to

$$\begin{pmatrix} \sin\alpha_1 b & \sinh\alpha_2 b \\ \alpha_1\cos\alpha_1 b & \alpha_2\cosh\alpha_2 b \end{pmatrix} \begin{Bmatrix} B_1 \\ D_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{25}$$

For obtaining nontrivial solutions, the determinant of coefficients matrix of Eq. (25) for B_1, and D_1 must be zero, leading to the following eigenvalue equation with respect to y

$$\alpha_2\tan\alpha_1 b = \alpha_1\tanh\alpha_2 b. \tag{26}$$

From Eq. (25), we have

$$D_1 = -B_1\frac{\sin\alpha_1 b}{\sinh\alpha_2 b}. \tag{27}$$

Substituting Eqs. (24) and (27) into Eq. (17) and letting B_1=1, we can determine the normal symplectic eigenfunction $\bar{W}(y)$ as

$$\bar{W}(y) = \sin\alpha_1 y - \frac{\sin\alpha_1 b}{\sinh\alpha_2 b}\sinh\alpha_2 y. \tag{28}$$

For other cases, the eigenvalue equation and the normal eigenfunctions can be obtained similarly. The eigensolutions of Eq. (13) for three typical cases S-S, S-C and C-C with respect to edges $y = 0$ and $y = b$ are given in Table 1.

In Table 1, the coefficient k_1 is given by

$$k_1 = \frac{\cos\alpha_1 b - \cosh\alpha_2 b}{(\alpha_2/\alpha_1)\sin\alpha_1 b - \sinh\alpha_2 b}. \tag{29}$$

It is readily found that the eigenvalue equation in Table 1 can not be solved independently for both cases S-C and C-C, but can be solved independently for the case S-S, this is also the reason why the inverse method can be applied only to the case with at least two opposite edges simply supported.

For the other two opposite edges $x = 0$ and $x = a$, if the boundary conditions are also S-S, or S-C or C-C, the eigenvalue equation and the eigenfunction $\Phi(x)$ (see Eq. (20)) can be solved similarly and will not be described in detail here. Therefore the natural mode in Eq. (19) is determined corresponding to $\Phi(x)$ and $\bar{W}(y)$. For cases SSCC, SCCC and CCCC as shown in Fig. 2, the exact normal modes and eigenvalue equations are summarized in Table 2, which are unable to be obtained by the inverse method.

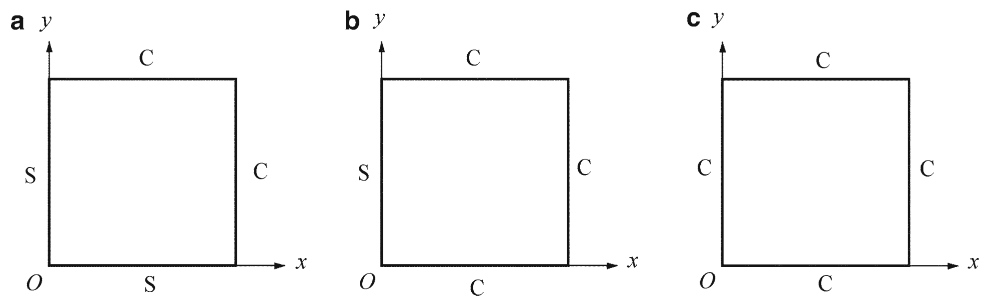

Fig. 2 The boundary conditions of plate. **a** SSCC; **b** SCCC; **c** CCCC

Table 2 The normal modes and eigenvalue equations for the cases SSCC, SCCC and CCCC

SSCC	Eigenvalue equations $\beta_2 \tan \beta_1 a = \beta_1 \tanh \beta_2 a, \quad \alpha_2 \tan \alpha_1 b = \alpha_1 \tanh \alpha_2 b$ Normal mode $W(x, y) = \left(\sin \beta_1 x - \frac{\sin \beta_1 a}{\sinh \beta_2 a} \sinh \beta_2 x\right) \cdot \left(\sin \alpha_1 y - \frac{\sin \alpha_1 b}{\sinh \alpha_2 b} \sinh \alpha_2 y\right)$
SCCC	Eigenvalue equations $\beta_2 \tan \beta_1 a = \beta_1 \tanh \beta_2 a, \quad \frac{1 - \cos \alpha_1 b \cosh \alpha_2 b}{\sin \alpha_1 b \sinh \alpha_2 b} = \frac{\alpha_1^2 - \alpha_2^2}{2\alpha_1 \alpha_2}$ Normal mode $W(x, y) = \left(\sin \beta_1 x - \frac{\sin \beta_1 a}{\sinh \beta_2 a} \sinh \beta_2 x\right) \cdot \left(-\cos \alpha_1 y + \frac{\alpha_2}{\alpha_1} k_1 \sin \alpha_1 y + \cosh \alpha_2 y - k_1 \sinh \alpha_2 y\right)$
CCCC	Eigenvalue equations $\frac{1 - \cos \beta_1 a \cosh \beta_2 a}{\sin \beta_1 a \sinh \beta_2 a} = \frac{\beta_1^2 - \beta_2^2}{2\beta_1 \beta_2}, \quad \frac{1 - \cos \alpha_1 b \cosh \alpha_2 b}{\sin \alpha_1 b \sinh \alpha_2 b} = \frac{\alpha_1^2 - \alpha_2^2}{2\alpha_1 \alpha_2}$ Normal mode $W(x, y) = \left(-\cosh \beta_1 x + \frac{\beta_2}{\beta_1} k_2 \sinh \beta_1 x + \cos \beta_2 x - k_2 \sin \beta_2 x\right) \cdot \left(-\cos \alpha_1 y + \frac{\alpha_2}{\alpha_1} k_1 \sin \alpha_1 y + \cosh \alpha_2 y - k_1 \sinh \alpha_2 y\right),$ $k_2 = \frac{\cos \beta_1 a - \cosh \beta_2 a}{(\beta_2/\beta_1) \sin \beta_1 a - \sinh \beta_2 a}$

In Table 2, the two eigenvalue equations for any case must be solved together with one of the following two equations

$$\beta_1^2 + \beta_2^2 = 2k^2, \tag{30}$$

$$\alpha_1^2 + \alpha_2^2 = 2k^2, \tag{31}$$

which can be derived from Eq. (16b) and Eq. (16a), respectively.

5 Numerical examples

Consider a rectangular thin plate with $h = 0.02\,\text{m}$, $\rho = 2{,}800\,\text{kg}\,\text{m}^{-3}$, $E = 72 \times 10^9\,\text{Pa}$, $\upsilon = 0.3$, and $a \times b = 1\,\text{m} \times 1.2\,\text{m}$. The Newton method is adopted to solve the transcendental eigenvalue equations. In Table 3, Table 4 and Table 5, ka denotes the present exact dimensionless

Table 3 The first ten frequencies for case SSCC

	1	2	3	4	5	6	7	8	9	10
$\beta_1 a$	3.72	3.52	7.00	6.86	3.41	10.18	6.74	10.09	3.35	6.66
ka	4.78	6.76	7.55	8.87	9.11	10.55	10.72	11.51	11.59	12.86
ka*	4.79	6.77	7.55	8.87	9.11	10.55	10.73	11.51	11.59	12.87

Table 4 The first ten frequencies for case SCCC

	1	2	3	4	5	6	7	8	9	10
$\beta_1 a$	3.68	3.49	6.99	6.84	3.39	10.17	6.73	10.08	3.34	9.99
ka	5.05	7.24	7.64	9.13	9.67	10.59	11.11	11.65	12.18	13.21
ka*	5.08	7.26	7.65	9.14	9.68	10.59	11.12	11.66	12.19	13.22

Table 5 The first ten frequencies for case CCCC

	1	2	3	4	5	6	7	8	9	10
$\beta_1 a$	4.33	3.90	7.72	7.45	3.68	10.93	7.21	10.77	3.56	7.04
ka	5.49	7.42	8.30	9.56	9.76	11.31	11.39	12.23	12.23	13.52
ka*	5.54	7.45	8.31	9.59	9.78	11.32	11.41	12.23	12.26	13.54

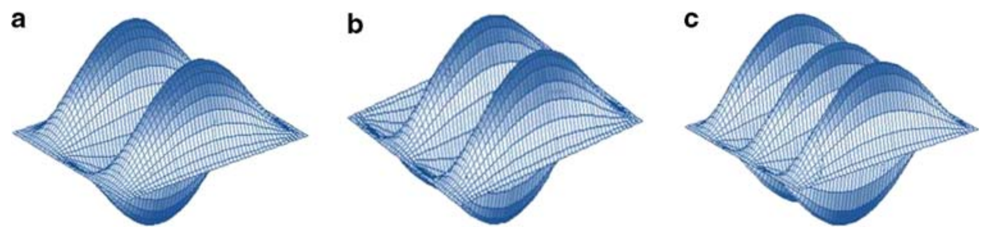

Fig. 3 Graphic illustration of the three exact mode shapes of case CCCC. **a** The 5th mode shape; **b** The 9th mode shape; **c** The 13th mode shape

frequencies, and ka* denotes the results of FEM, which are calculated by MSC Nastran with mesh 100 × 120 and Bending Panel element. It follows from Table 3, Table 4 and Table 5 that the results of the two methods are in close agreement, and thus the exact frequency equations and normal modes proposed in this paper are validated. The mode shapes given by the exact solution are the same as those obtained by FEM, among which only three mode shapes of the case CCCC are shown in Fig. 3.

6 Conclusions

Hamiltonian symplectic dual method and the separation of variables are adopted to solve the transverse free vibration problems of rectangular thin plates. The general mathematical expression of natural mode is derived, which satisfies the governing equation of the eigenvalue problem exactly and holds true for all kinds of boundary conditions.

The exact normal modes and frequency equations can be obtained for any combination of separable simply-supported and clamped boundary conditions. The exact eigensolutions for the three cases SSCC, SCCC and CCCC presented in this paper have not been available in the previous literature.

The application of the Hamiltonian symplectic dual method and the separation of spatial variables are not limited to the rectangular thin plate, and can be extended to the static and dynamic problems of planes, thick plates and others.

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