A theory of hyperelasticity of multi-phase media with surface/interface energy effect

Z. P. Huang, J. Wang

LTCS and Department of Mechanics and Engineering Science, Peking University, Beijing 100871, P. R. China (Email: huangzp@pku.edu.cn, Tel: +86 10 6276 7170, Fax: + 86 10 6275 7948)

Summary. In addition to the classical governing equations in continuum mechanics, two kinds of governing equation are necessary in the solution of boundary-value problems for the stress fields in multi-phase hyperelastic media with the surface/interface energy effect. The first is the interface constitutive relation, and the second is the discontinuity conditions of the traction across the interface, namely, the Young-Laplace equations. In this paper, the interface constitutive relations are presented in terms of the interface energy in both Lagrangian and Eulerian descriptions within the frame work of finite deformation, and the expressions of the interface stress for an isotropic interface are given as a special case. Then, by introducing a fictitious stress-free configuration, a new energy functional for multi-phase hyperelastic media with interface energy effect is proposed . The functional takes into account the interface energy and the interface stress-induced "residual" elastic field, which reflects the intrinsic physical properties of the material. All field equations, including the generalized Young-Laplace equation, can be derived from the stationary condition of this functional. The present theory is illustrated by simple examples. The results in this paper provide a theoretical framework for studying the elastostatic problems of multi-phase hyperelastic bodies that involve surface/interface energy effect at finite deformation.

1 Introduction

The physical origin of surface/interface energy and surface/interface stresses in solids has been discussed by many researchers (e.g. cf. [1, 2, 3, 4, 5, 6], among others). At the atomic scale, the microscopic structures of a surface of a body or an interface between a pair of bodies are quite different from those associated with its interiors. This can be described macroscopically by the excess free energy of a surface/interface and the corresponding surface/interface stress. The surface/interface free energy is usually defined as a reversible work per unit area involved in creating a new surface/interface. This excess free energy will generally vary when the surface/interface is deformed. The surface/interface stress in solids is defined through the change in excess free energy when the surface/interface is stretched elastically. From the above definitions it can be seen that the surface/interface energy and surface/interface stress have different nature, although for a liquid, the magnitudes of the surface energy and the surface stress (surface tension) are the same. In the following, for expediency, the discussion will be referenced to an interface; it is equally applicable to an surface.

The classical continuum mechanics is generally applied to the study of the deformation of materials at the macroscale where the ratio of the interface region to the bulk is negligibly small, so the interface effect can be rightfully neglected. However, many studies show that for nanostructured materials and heterogeneous materials containing inhomogeneities of the nanosize, the interface effect will become significant and should be taken into account (e.g. [7, 8, 9, 10, 11], etc.). It should be mentioned that previously, Gurtin and Murdoch [12] developed a mathematical framework for studying the mechanical behaviour of solids taking into account the surface stress within the formalism of continuum mechanics. They also presented a linearized surface stress-strain constitutive relation. Steigmann and Ogden [13] generalized the Gurtin-Murdoch theory [12] to account for the effect of the flexural resistance of elastic films attached to the bounding surfaces of solids. In this paper, we shall developed a theoretical framework which gives all the governing equations for solving the elastostatic problems of multi-phase hyperelastic media at finite deformation. The major differences from the previous works in the literature are that, first, a fictitious configuration is introduced to facilitate the study of the interface energy effect, and its physical background and the necessity are delineated, and second, a new energy functional is presented and it is shown that the governing equations, including the generalized Young-Laplace equations, can be obtained from the stationary conditions of this functional. In addition, some example problems of hyperelasticity are solved to demonstrate the applicability of the theoretical framework.

2 The constitutive relations of the interface

Consider a multi-phase hyperelastic solid in 3-dimensional Euclidean space. The multi-phase body occupies a domain V_0 with a boundary ∂V_0 when it is neither subjected to any body force nor to any external surface traction on its boundary. We refer to this configuration as the initial configuration, denoted by $\mathcal{K}_0$. Although, in continuum mechanics, any configuration can be chosen as a reference configuration, yet in order to simplify the discussion, this initial configuration $\mathcal{K}_0$ will be specifically taken as the reference configuration. When the body is subjected to external loading, which can be a body force and/or a traction/displacement on its boundary, the deformation of the body can be represented by the following mapping:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \qquad \mathbf{X} \in V_0, \tag{1}$$

where $\mathbf{X}$ and $\mathbf{x}$ represent the position vectors of a typical material particle before and after the deformation, respectively. Then the deformation gradient $\mathbf{F}$ satisfies $\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}$. Assume that the interfaces (including surfaces) in the reference configuration $\mathcal{K}_0$ are smooth and closed surfaces, denoted collectively by A_0. In a curvilinear coordinate system, an interface can be parametrized locally by coordinates θ^α $(\alpha = 1, 2)$. The position function $\mathbf{Y}$ of the interface is identified with the restriction of $\mathbf{X}$ to A_0: $\mathbf{Y}(\theta^{\mathbf{1}}, \theta^{\mathbf{2}}) = \mathbf{X}\mid_{\mathrm{on\,A_0}}$. The corresponding covariant base vectors at $\mathbf{Y}$ are $\mathbf{A}_\alpha = \mathbf{Y}_{,\alpha}$ $(\alpha = 1, 2)$, with the unit normal vector to the surface A_0 denoted by $\mathbf{A}_3$. Here, and in the following, the Greek indices run from 1 to 2, and the Latin indices run from 1 to 3. After undergoing a deformation, the point $\mathbf{Y}$ on A_0 will move to point $\mathbf{y}$ on surface A in the current configuration, denoted by $\mathcal{K}$. We assume that this surface A is convected by the deformation of the bulk solid so that it can be expressed by the parametric equation $\mathbf{y}(\theta^1, \theta^2) = \chi(\mathbf{Y}(\theta^{\mathbf{1}}, \theta^{\mathbf{2}}))$, in which θ^α are regarded as identifying the same particle before and after deformation. In a rectangular Cartesian coordinate with base vector $\mathbf{e}_i$ $(i = 1, 2, 3)$, the above equation can be written as $\mathbf{y} = y_i(\theta^1, \theta^2)\mathbf{e}_i$. Then the corresponding covariant base vectors on surface A are $\mathbf{a}_\alpha = \mathbf{y}_{,\alpha} = y_{i,\alpha}\,\mathbf{e}_i$ with the normal vector $\mathbf{a}_3$. For the surface A_0, we can define a tangent plane $\mathcal{J}^0_{\mathbf{Y}}$ at $\mathbf{Y} \in A_0$ so that for a given $\mathbf{Y} \in A_0$, there is a linear transform that maps $\mathcal{J}^0_{\mathbf{Y}}$ into the tangent plane $\mathcal{J}_{\mathbf{y}}$ at $\mathbf{y} \in A$. This mapping can be expressed by the deformation gradient $\mathbf{F}_s = \mathbf{a}_\alpha \bigotimes \mathbf{A}^\alpha$ in a 2-dimensional space, where $\mathbf{A}^\alpha$ are the contravariant base vectors at $\mathbf{Y} \in A_0$. Then we can define the right and left Cauchy-Green tensors of the interface $\mathbf{C}_s = \mathbf{F}_s^T \cdot \mathbf{F}_s$ and $\mathbf{B}_s = \mathbf{F}_s \cdot \mathbf{F}_s^T$. As the symmetric $\mathbf{C}_s$ and $\mathbf{B}_s$ are positive-definite, we can further define the right and left stretch tensors of the interface $\mathbf{U}_s = \mathbf{C}_s^{1/2}$ and $\mathbf{V}_s = \mathbf{B}_s^{1/2}$.

The spectral decompositions of $\mathbf{U}_s$ and $\mathbf{V}_s$ can be written as

$$\mathbf{U}_s = \sum_{\alpha=1}^{2} \lambda_\alpha \mathbf{U}_\alpha \otimes \mathbf{U}_\alpha \quad , \quad \mathbf{V}_s = \sum_{\alpha=1}^{2} \lambda_\alpha \mathbf{V}_\alpha \otimes \mathbf{V}_\alpha \tag{2}$$

where $\lambda_\alpha > 0$ $(\alpha = 1, 2)$ are the principal stretches on the interface, $\mathbf{U}_\alpha$ and $\mathbf{V}_\alpha = \mathbf{R}_s \cdot \mathbf{U}_\alpha$ $(\alpha = 1, 2)$ are the (unit) eigenvectors of $\mathbf{U}_s$ and $\mathbf{V}_s$ respectively, and $\mathbf{R}_s$ is the proper orthogonal tensor in the polar decomposition of $\mathbf{F}_s$: $\mathbf{F}_s = \mathbf{R}_s \cdot \mathbf{U}_s = \mathbf{V}_s \cdot \mathbf{R}_s$. Following the discussion of Seth [14], the

Lagrangian strain tensors of the interface can be defined by

$$\mathbf{E}_s^{(m)} = \frac{1}{2m}\left(\mathbf{U}_s^{2m} - \mathbf{1}_0\right), \, m \neq 0$$
$$\mathbf{E}_s^{(0)} = \ln \mathbf{U}_s \, , \, m = 0 \tag{3}$$

where m is a real number, and $\mathbf{1}_0$ is the second-rank identity tensor in tangent plane $\mathcal{J}_{\mathbf{Y}}^0$ in the reference configuration $\mathcal{K}_0$. The Eulerian strain tensors which are based on $\mathbf{V}_s$ may also be constructed similarly.

The constitutive relations of the interface have been extensively studied by many researchers in the literature. But most of works on this subject are confined to the infinitesimal deformation approximations. Let the excess free energy of the interface per unit area of A in the current configuration be denoted by γ, which not only depends on the particle coordinates (θ^1, θ^2), but also on the interface strain. For the sake of notational simplicity, the dependance of γ on (θ^1, θ^2) will be suppressed in the following. Then the interface energy per unit area of A_0 in the reference configuration can be written as $J_2\gamma$, where $J_2 = \det \mathbf{U}_s$ is the ratio between the area elements dA and dA_0, i.e. $dA = J_2 dA_0$. In the course of deformation, the variation in the excess free energy of the interface on the area element dA can be written as $\delta(\gamma dA) = \delta(J_2\gamma)dA_0$, which, according to Gibbs [1], should be the reversible work needed to elastically stretch this pre-existing surface element

$$\delta\left(J_2\gamma\right)\mathrm{d}A_0 = \left(\mathbf{T}_s^{(m)} : \delta\mathbf{E}_s^{(m)}\right)\mathrm{d}A_0 \tag{4}$$

where $\mathbf{T}_s^{(m)}$ is the interface stress conjugate to $\mathbf{E}_s^{(m)}$.

In the case of small deformation, $\mathbf{E}_s^{(m)}$ and $\mathbf{T}_s^{(m)}$ can be replaced by the infinitesimal interfacial strain $\boldsymbol{\varepsilon}_s$ and the corresponding interfacial stress $\boldsymbol{\tau}_s$, respectively. Furthermore, for small deformation, we have

$$J_2 \doteq 1 \, , \, \delta J_2 = \frac{\partial J_2}{\partial \boldsymbol{\varepsilon}_s} : \delta\boldsymbol{\varepsilon}_s \doteq \mathrm{tr}\left(\delta\boldsymbol{\varepsilon}_s\right) = \mathbf{1} : \delta\boldsymbol{\varepsilon}_s \tag{5}$$

where $\mathbf{1}$ is the second-rank identity tensor in two-dimensional space $\mathcal{J}_y$. Hence from Eq. (4), we obtain

$$\boldsymbol{\tau}_s = \gamma\mathbf{1} + \frac{\partial\gamma}{\partial\boldsymbol{\varepsilon}_s} \tag{6}$$

This is just the venerable Shuttleworth equation (e.g. [4, 5, 6]).

A linearized constitutive relation of the interface has been given by Gurtin and Murdoch [12]. For an isotropic interface relative to the reference configuration, the Cauchy stress of the interface was suggested to be

$$\boldsymbol{\tau}_s = \gamma_0^*\mathbf{1} + \lambda_s\left(\mathrm{tr}\boldsymbol{\varepsilon}_s\right)\mathbf{1} + 2\mu_s\boldsymbol{\varepsilon}_s \tag{7}$$

where $\gamma_0^*\mathbf{1}$ is the residual interface stress at $\mathcal{K}_0$, λ_s and μ_s are Lame constants of the interface. From the above equation, a linearized constitutive relation for the interfacial Piola-Kirchhoff stress of the first kind can be written as

$$\mathbf{S}_s^{(L)} = \gamma_0^*\mathbf{1} + \left(\lambda_s + \gamma_0^*\right)\left(\mathrm{tr}\boldsymbol{\varepsilon}_s\right)\mathbf{1} + 2\left(\mu_s - \gamma_0^*\right)\boldsymbol{\varepsilon}_s + \gamma_0^*\mathbf{u}\nabla_{0s} \tag{8}$$

where $\mathbf{u}\nabla_{0s}$ is the displacement gradient of the interface at $\mathcal{K}_0$.

The derivation of the constitutive relation of the interface at finite deformation from Eq. (4) is straightforward, which gives

$$\mathbf{T}_s^{(m)} = \frac{\partial\left(J_2\gamma\right)}{\partial\mathbf{E}_s^{(m)}} \tag{9}$$

where the Lagrangian description has been used and $J_2\gamma$ can be regarded as a potential function of the Lagrangian strain of the interface. In particular, the Piola–Kirchhoff stresses of the first and second kind of the interface are given by

$$\mathbf{S}_s = 2\mathbf{F}_s \cdot \frac{\partial(J_2\gamma)}{\partial\mathbf{C}_s}, \quad \mathbf{T}_s^{(1)} = 2\frac{\partial(J_2\gamma)}{\partial\mathbf{C}_s}. \tag{10}$$

In the Eulerian description, the Cauchy stress of the interface can be expressed as

$$\boldsymbol{\sigma}_s = \frac{1}{J_2}\mathbf{F}_s \cdot \mathbf{T}_s^{(1)} \cdot \mathbf{F}_s^T. \tag{11}$$

It should be noted that the above expressions are for anisotropic interface which depend on the crystallographic parameters of the interfaces in solids. If the interface is assumed to be isotropic relative to $\mathcal{K}_0$, i.e., the underlying reference configuration is a undistorted state, then γ can be expressed as a function of the invariants of $\mathbf{U}_s$ or $\mathbf{V}_s$: $\gamma = \gamma(J_1, J_2)$, where $J_1 = \mathrm{tr}\mathbf{U}_s = \mathrm{tr}\mathbf{V}_s, J_2 = \det\mathbf{U}_s = \det\mathbf{V}_s$ are the first and second invariants of $\mathbf{U}_s$ (or $\mathbf{V}_s$). Noting that $\frac{\partial J_1}{\partial \mathbf{C}_s} = \frac{1}{2}\mathbf{U}_s^{-1}, \frac{\partial J_2}{\partial \mathbf{C}_s} = \frac{1}{2}J_2\mathbf{C}_s^{-1}$, we obtain

$$\mathbf{T}_s^{(1)} = J_2\left[\frac{\partial\gamma}{\partial J_1}\mathbf{U}_s^{-1} + (J_2\frac{\partial\gamma}{\partial J_2} + \gamma)\mathbf{C}_s^{-1}\right], \tag{12}$$

The engineering and the logarithmic stresses of the interface can be written as

$$\mathbf{T}_s^{(1/2)} = J_2\left[\frac{\partial\gamma}{\partial J_1}\mathbf{1}_0 + \left(J_2\frac{\partial\gamma}{\partial J_2} + \gamma\right)\mathbf{U}_s^{-1}\right] \tag{13}$$

$$\mathbf{T}_s^{(0)} = J_2\left[\frac{\partial\gamma}{\partial J_1}\mathbf{U}_s + \left(J_2\frac{\partial\gamma}{\partial J_2} + \gamma\right)\mathbf{1}_0\right] \tag{14}$$

and the Cauchy stress of the interface is

$$\boldsymbol{\sigma}_s = \frac{\partial\gamma}{\partial J_1}\mathbf{V}_s + (J_2\frac{\partial\gamma}{\partial J_2} + \gamma)\mathbf{1}(\mathbf{y}), \tag{15}$$

where $\mathbf{1}(\mathbf{y})$ is the second-rank unit tensor in 2-dimensional space $\mathcal{J}_y$. Especially, when the current configuration $\mathcal{K}$ and the reference one $\mathcal{K}_0$ coincide, the Cauchy stress at the reference configuration can be expressed by $\boldsymbol{\sigma}_s^* = \gamma_0^*\mathbf{1}$, where $\gamma_0^* = \gamma_0 + \gamma_1 + \gamma_2$ is the residual interface tension. $\gamma_1 = \frac{\partial\gamma}{\partial J_1}|_{J_1=2,J_2=1}$, and $\gamma_2 = \frac{\partial\gamma}{\partial J_2}|_{J_1=2,J_2=1}$ reflect the nature of solids, whereas $\gamma_0 = \gamma(2,1)$ reflects the nature of liquids. γ_0, γ_1 and γ_2 represent the intrinsic physical properties of the interface, and they are, and should be, determined by the joining materials and the adhering condition.

3 "Residual" elastic field induced by interface energy

In a multi-phase hyperelastic body, the stress field cannot be determined by the deformation gradient $\mathbf{F}$ only. The reason for this is as follows. Consider two materials that are bonded to form an interface in the multi-phase body, and assume that they are immiscible, that is, either atomic diffusion or the formation of defects across the interface can be neglected, so that the atoms at the interface A_0 remain at the interface A after the deformation. Because the two materials have different internal micro-structures (on the atomic - molecular level), the micro-structure of the interface will be different from those of the interior of the two materials when they are conglutinated together [15]. Thus, like the case of a surface, the conglutination generates not only the excess interface free energy, but also an interface stress. These quantities are generally not zero in the initial reference configuration $\mathcal{K}_0$ that is not subjected to any external loading. We refer the interface stress in this reference configuration $\mathcal{K}_0$ as the "residual" interface stress. The elastic field which is induced by the interface stress is referred to as the "residual" elastic field. It should be pointed out that the terminology "residual" stress has been extensively used in the literature. Here, the "residual" elastic field is different from those both in plasticity and in continuum theory of distributed dislocations. In plasticity, the residual stress field is induced by the plastic strain, which is incompatible. The strain field in continuum theory of distributed dislocation is also incompatible. However, the "residual" elastic field in the present paper is compatible, which is entirely due to the existence of the surface/interface energy and the surface/interface stress. To facilitate the description of the deformation problem of the multi-phase material, we can hypothetically split the body into homogeneous pieces along the interfaces, and imaginarily let them return to their stress-free state. It should be emphasized that in general, this process cannot be fulfilled in reality, since it is assumed that the micro-structures (at atomic level) of the surfaces, which are obtained by debonding the interfaces, would have the same micro-structures as those of their respective interior matrices. Thus the above splitting is only an imaginary operation. The configuration so obtained will be called the "fictitious stress-free configuration" $\mathcal{K}_*$. This "fictitious stress-free configuration" will not exist in general, because splitting a body will inevitably create new surfaces, and the new surface energy will in turn induce new residual stress field. If the two surfaces that are created by debonding an interface A_0 are A_{*1} and A_{*2}, and the densities of the two materials, herein referred to as material 1 and material 2, are denoted by ρ_{*1} and ρ_{*2}, respectively, then the above surfaces A_{*1} and A_{*2} should be regarded as having the interior micro-structures as if they were in materials 1 and 2, respectively. The above splitting can be considered to be a mapping that maps an interface into two surfaces. Then a point P on A_0 will be mapped into two points P_1 and P_2 on the surfaces of the two materials so obtained. If the corresponding unit normal vectors at P_1 and P_2 are denoted by ${}_*\mathbf{N}_1$ and ${}_*\mathbf{N}_2$, respectively, then, generally, they will not be collinear. Let a representative material particle, denoted by $\mathbf{X}$, in the reference configuration $\mathcal{K}_0$ be moved to $\mathbf{X}_*$ in the fictitious stress-free configuration $\mathcal{K}_*$. Then the deformation gradient tensors $\mathbf{F}_1^*$ for material 1 and $\mathbf{F}_2^*$ for material 2 satisfy the following relation:

$$\begin{aligned} \mathbf{A}_3\, dA_0 &= (\det\mathbf{F}_1^*)(\mathbf{F}_1^*)^{-T} \cdot {}_*\mathbf{N}_1\, dA_{*1} \\ &= (\det\mathbf{F}_2^*)(\mathbf{F}_2^*)^{-T} \cdot {}_*\mathbf{N}_2\, dA_{*2}, \end{aligned} \tag{16}$$

where dA_{*1} and dA_{*2} are the area elements on the surfaces A_{*1} and A_{*2} that correspond to dA_0 on A_0.

The above discussion indicates that due to the existence of the residual elastic field in $\mathcal{K}_0$, the elastic field in the multi-phase hyperelastic body when subjected to external loading is not determined by the deformation gradient $\mathbf{F}$ only, as mentioned before. Instead, it should depend upon the deformation gradient $\mathbf{F}\cdot\mathbf{F}^*$. Thus the hyperelastic potential ψ_0 should be expressed as a function of position vector $\mathbf{X}$ and $\tilde{\mathbf{C}} = (\mathbf{F}\cdot\mathbf{F}^*)^T \cdot (\mathbf{F}\cdot\mathbf{F}^*)$. For simplicity, the dependence of ψ_0 on $\mathbf{X}$ will be suppressed. Hence the Piola-Kirchhoff stresses of the first and second kind corresponding to the "fictitious stress-free" configuration $\mathcal{K}_*$ and the reference configuration $\mathcal{K}_0$ are, respectively,

$$\mathbf{S}^* = 2\rho_*\mathbf{F}\cdot\mathbf{F}^*\cdot\frac{\partial\psi_0}{\partial\tilde{\mathbf{C}}}, \qquad \mathbf{T}^* = 2\rho_*\frac{\partial\psi_0}{\partial\tilde{\mathbf{C}}}, \tag{17}$$

$$\mathbf{S}^0 = 2\rho_0\mathbf{F}\cdot\mathbf{F}^*\cdot\frac{\partial\psi_0}{\partial\tilde{\mathbf{C}}}\cdot\mathbf{F}^{*T}, \qquad \mathbf{T}^0 = 2\rho_0\,\mathbf{F}^*\cdot\frac{\partial\psi_0}{\partial\tilde{\mathbf{C}}}\cdot\mathbf{F}^{*T}. \tag{18}$$

The Cauchy stress corresponding to the configuration $\mathcal{K}$ is

$$\boldsymbol{\sigma} = 2\rho\,\mathbf{F}\cdot\mathbf{F}^*\cdot\frac{\partial\psi_0}{\partial\tilde{\mathbf{C}}}\cdot\mathbf{F}^{*T}\cdot\mathbf{F}^T. \tag{19}$$

In the above expressions, ρ_*, ρ_0 and ρ denote the densities of the body in the configurations $\mathcal{K}_*$, $\mathcal{K}_0$ and $\mathcal{K}$, respectively. Conservation of mass yields $\rho_*\, dv_* = \rho_0\, dv_0 = \rho\, dv$.

4 A new energy functional and the generalized Young-Laplace equations

This section discusses the equilibrium equations both in the bulk solid and on the surface/interface. These equations can be derived from the stationary condition of a new energy functional, which consists of two parts. The first part is the total free energy, and the second is the potential of external loading. As was mentioned before, even there is no external loading, there may exist an elastic field induced by the interface energy. Therefore, the hyperelastic potential ψ_0 depends not only on $\mathbf{C}$, but also on $\mathbf{F}^*$, i.e., ψ_0 is a function of $\tilde{\mathbf{C}}$. Hence for a multi-phase hyperelastic body, we have the following proposition:

Proposition

Of all admissible displacement fields $\mathbf{u}$ satisfying a prescribed displacement $\bar{\mathbf{u}}_0$ on the boundary ∂V_{0u}, the following functional assumes a stationary value when $\mathbf{u}$ is that of the equilibrium state under the action of a body force $\rho_0\mathbf{f}$ in V_0 and a traction $\bar{\mathbf{t}}_0$ on its boundary ∂V_{0T}:

$$\begin{aligned}\Pi(\mathbf{u}) &= \int_{A_0} J_2\,\gamma(\mathbf{C}_s)\,dA_0 + \int_{V_0} \rho_0\psi_0(\tilde{\mathbf{C}})\,dv_0 \\ &\quad - \int_{V_0} \rho_0\mathbf{f}\cdot\mathbf{u}\,dv_0 - \int_{\partial V_{0T}} \bar{\mathbf{t}}_0\cdot\mathbf{u}\,ds_0. \end{aligned} \tag{20}$$

It should be noted that the displacement $\mathbf{u}$ in the above expression is from the reference configuration $\mathcal{K}_0$ to the current configuration $\mathcal{K}$, and when $\mathbf{u}=\mathbf{0}$, i.e., $\mathbf{C}_s = \mathbf{1_0}$, the interface energy $\gamma(\mathbf{1})$ is not zero, which is an intrinsic physical attribute and should be determined by the joining materials and the adhering condition, as mentioned before.

Proof

Let the displacement field $\mathbf{u}$ be subjected to a variation $\delta\mathbf{u}$. Then the first term on the right hand side of Eq. (20) can be expressed as

$$\begin{aligned}\int_{A_0} \delta(J_2\,\gamma)\,dA_0 &= \int_{A_0} \mathbf{S}_s : (\delta\mathbf{u}\,\nabla_0)\,dA_0 \\ &= \int_{A} \boldsymbol{\sigma}_s : (\delta\mathbf{u}\,\nabla)\,dA, \end{aligned} \tag{21}$$

where ∇_0 and ∇ are the gradient operators in the reference and current configurations, respectively.

$\delta\mathbf{u}$ can be decomposed into a vector $\delta\mathbf{u}_{0s}$ in the tangent plane $\mathcal{J}^0_{\mathbf{Y}}$ relative to A_0 and a vector $\delta\, u^n_0\mathbf{A}_3$ along the normal direction of this tangent plane in the reference configuration $\mathcal{K}_0$; it can also be decomposed into $\delta\mathbf{u}_s$ and $\delta\, u^n\mathbf{a}_3$ in, and normal to, the tangent plane $\mathcal{J}_{\mathbf{y}}$ relative to A in the current configuration $\mathcal{K}$.

Using the Weingarten formula and noting that $\mathbf{S}_s$ and $\boldsymbol{\sigma}_s$ are tensors in the 2-dimensional tangent planes $\mathcal{J}^0_{\mathbf{Y}}$ and $\mathcal{J}_{\mathbf{y}}$, respectively, we have

$$\mathbf{S}_s : (\delta\mathbf{u}\nabla_0) = (\delta\mathbf{u}_{0s}\cdot\mathbf{S}_s)\cdot\nabla_{0s} - \delta\mathbf{u}_{0s}\cdot(\mathbf{S}_s\cdot\nabla_{0s}) - \delta u^n_0(\mathbf{S}_s : \mathbf{b}_0), \tag{22}$$

and

$$\boldsymbol{\sigma}_s : (\delta\mathbf{u}\nabla) = (\delta\mathbf{u}_s\cdot\boldsymbol{\sigma}_s)\cdot\nabla_s - \delta\mathbf{u}_s\cdot(\boldsymbol{\sigma}_s\cdot\nabla_s) - \delta u^n(\boldsymbol{\sigma}_s : \mathbf{b}), \tag{23}$$

where ∇_{0s} and ∇_s are the gradient operators on the surfaces A_0 and A, respectively, namely,

$$(\delta\mathbf{u}_{0s})\nabla_{0s} = \delta u^\alpha_{0s}|_\beta \mathbf{A}_\alpha \bigotimes \mathbf{A}^\beta, \tag{24}$$

$$(\delta\mathbf{u}_s)\nabla_s = \delta u^\alpha_s|_\beta \mathbf{a}_\alpha \bigotimes \mathbf{a}^\beta, \tag{25}$$

$\delta u^\alpha_{0s}|_\beta$ and $\delta u^\alpha_s|_\beta$ are the surface covariant derivatives on A_0 and A, respectively. $\mathbf{b}_0 = b^\alpha_{0\,\beta}\mathbf{A}_\alpha \bigotimes \mathbf{A}^\beta$ and $\mathbf{b} = b^\alpha_{\ \beta}\mathbf{a}_\alpha \bigotimes \mathbf{a}^\beta$ are the second fundamental forms of the surfaces A_0 and A, respectively, namely, the curvature tensors.

Consider a region Ω_0 enclosed by an arbitrary closed smooth curve $\partial\Omega_0$ in the curved surface A_0. By using the Green-Stokes theorem, we have

$$\begin{aligned}\int_{A_0} \mathbf{S}_s : (\delta\mathbf{u}\nabla_0)\,dA_0 &= \int_{\partial\Omega_0} \delta\mathbf{u}_{0s}\cdot[\![\mathbf{S}_s]\!]\cdot\mathbf{n}_0\,dl_0 \\ &\quad - \int_{A_0} \left[\delta\mathbf{u}_{0s}\cdot(\mathbf{S}_s\cdot\nabla_{0s}) + \delta u^n_0(\mathbf{S}_s : \mathbf{b}_0)\right] dA_0, \end{aligned} \tag{26}$$

where dl_0 is the element of the arclength on $\partial\Omega_0$. $\mathbf{n}_0 = \mathbf{l}_0 \times \mathbf{A}_3$ is the unit normal vector of the curve $\partial\Omega_0$ with $\mathbf{l}_0$ being the tangent vector of $\partial\Omega_0$. $[\![\mathbf{S}_s]\!]$ is the discontinuity of $\mathbf{S}_s$ across the curve $\partial\Omega_0$. Similarly, for a region Ω enclosed by $\partial\Omega$ in the surface A, we have

$$\int_A \boldsymbol{\sigma}_s : (\delta\mathbf{u}\nabla)\, dA = \int_{\partial\Omega} \delta\mathbf{u}_s \cdot [\![\boldsymbol{\sigma}_s]\!] \cdot \mathbf{n}\, dl \\ - \int_A \left[\delta\mathbf{u}_s \cdot (\boldsymbol{\sigma}_s \cdot \nabla_s) + \delta u^n (\boldsymbol{\sigma}_s : \mathbf{b})\right] dA, \tag{27}$$

where $\mathbf{n} = \mathbf{l} \times \mathbf{a}_3$, with $\mathbf{l}$ being the unit tangent vector of $\partial\Omega$. dl is the element of the arclength on $\partial\Omega$. $[\![\boldsymbol{\sigma}_s]\!]$ is the discontinuity of $\boldsymbol{\sigma}_s$ across the curve $\partial\Omega$.

Next, we consider the variation of the second term on the right hand side of Eq. (20). Noting that

$$\begin{aligned} \rho_0 \frac{\partial \psi_0(\tilde{\mathbf{C}})}{\partial \mathbf{C}} : \delta\mathbf{C} &= \rho_0 \left(\mathbf{F}^* \cdot \frac{\partial \psi_0}{\partial \tilde{\mathbf{C}}} \cdot \mathbf{F}^{*T} \right) : \delta\mathbf{C} \\ &= \mathbf{S}^0 : (\delta\mathbf{u}\nabla_0) = (\det\mathbf{F})\, \boldsymbol{\sigma} : (\delta\mathbf{u}\nabla), \end{aligned} \tag{28}$$

and by conservation of mass, we have

$$\delta \int_{V_0} \rho_0 \psi_0(\tilde{\mathbf{C}})\, dv_0 = \int_{V_0} \mathbf{S}^0 : (\delta\mathbf{u}\nabla_0)\, dv_0 = \int_{\partial V_{0T}} \delta\mathbf{u} \cdot (\mathbf{S}^0 \cdot_0 \mathbf{N})\, ds_0 - \\ - \int_{A_0} \delta\mathbf{u} \cdot [\![\mathbf{S}^0]\!] \cdot \mathbf{A}_3\, dA_0 - \int_{V_0} \delta\mathbf{u} \cdot (\mathbf{S}^0 \cdot \nabla_0)\, dv_0. \tag{29}$$

Under Eulerian description, the above expressions can also be expressed as

$$\begin{aligned} \delta \int_{V_0} \rho_0 \psi_0(\tilde{\mathbf{C}})\, dv_0 &= \int_V \boldsymbol{\sigma} : (\delta\mathbf{u}\nabla)\, dv = \int_{\partial V_T} \delta\mathbf{u} \cdot (\boldsymbol{\sigma} \cdot \mathbf{N})\, ds \\ &\quad - \int_A \delta\mathbf{u} \cdot [\![\boldsymbol{\sigma}]\!] \cdot \mathbf{a}_3\, dA - \int_V \delta\mathbf{u} \cdot (\boldsymbol{\sigma} \cdot \nabla)\, dv. \end{aligned} \tag{30}$$

where ${}_0\mathbf{N}$ and $\mathbf{N}$ are unit normal vectors to the boundaries before and after the deformation, and $[\![\mathbf{S}^0]\!]$ and $[\![\boldsymbol{\sigma}]\!]$ are the discontinuities of $\mathbf{S}^0$ and $\boldsymbol{\sigma}$ across the surfaces A_0 and A, respectively.

Finally, the variations of the last two terms on the right hand side of Eq. (20) can be written as

$$- \int_{V_0} \rho_0 \delta\mathbf{u} \cdot \mathbf{f}\, dv_0 - \int_{\partial V_{0T}} \delta\mathbf{u} \cdot \bar{\mathbf{t}}_0\, ds_0. \tag{31}$$

When using the Eulerian description, the above expression is

$$- \int_V \rho \delta\mathbf{u} \cdot \mathbf{f}\, dv - \int_{\partial V_T} \delta\mathbf{u} \cdot \bar{\mathbf{t}}\, ds, \tag{32}$$

where the boundary traction $\bar{\mathbf{t}}_0$ in the reference configuration $\mathcal{K}_0$ and $\bar{\mathbf{t}}$ in the current configuration are related to each other by $\bar{\mathbf{t}}_0 ds_0 = \bar{\mathbf{t}}\, ds$.

From Eqs. (21)–(23), (26), (29) and (31), and the arbitrariness of $\delta\mathbf{u}$, it is seen that the vanishing of the variation of the functional in Eq. (20) is equivalent to the following conditions:

$$\mathbf{S}^0 \cdot \nabla_0 + \rho_0 \mathbf{f} = 0 \qquad (\text{in } V_0), \tag{33}$$

$$\mathbf{S}^0 \cdot {}_0\mathbf{N} = \bar{\mathbf{t}}_0 \qquad (\text{on } \partial V_{0T}), \tag{34}$$

$$\mathbf{A}_3 \cdot [\![\mathbf{S}^0]\!] \cdot \mathbf{A}_3 = -\mathbf{S}_s : \mathbf{b}_0 \qquad (\text{on } A_0), \tag{35}$$

$$\mathbf{P}_0 \cdot [\![\mathbf{S}^0]\!] \cdot \mathbf{A}_3 = -\mathbf{S}_s \cdot \nabla_{0s} \qquad (\text{on } A_0), \tag{36}$$

$$[\![\mathbf{S}_s]\!] \cdot \mathbf{n}_0 = 0 \qquad (\text{on curve } \partial\Omega_0 \text{ in } A_0), \tag{37}$$

where $\mathbf{P}_0 = \mathbf{I} - \mathbf{A}_3 \bigotimes \mathbf{A}_3$, and $\mathbf{I}$ is the second-rank unit tensor in 3-dimensional space. The corresponding equations under Eulerian description are

$$\boldsymbol{\sigma} \cdot \nabla + \rho \mathbf{f} = 0 \qquad (\text{in } V), \tag{38}$$

$$\boldsymbol{\sigma} \cdot \mathbf{N} = \bar{\mathbf{t}} \qquad (\text{on } \partial V_T), \tag{39}$$

$$\mathbf{a}_3 \cdot [\![\boldsymbol{\sigma}]\!] \cdot \mathbf{a}_3 = -\boldsymbol{\sigma}_s : \mathbf{b} \qquad (\text{on } A), \tag{40}$$

$$\mathbf{P} \cdot [\![\boldsymbol{\sigma}]\!] \cdot \mathbf{a}_3 = -\boldsymbol{\sigma}_s \cdot \nabla_s \qquad (\text{on } A), \tag{41}$$

$$[\![\boldsymbol{\sigma}_s]\!] \cdot \mathbf{n} = 0 \qquad (\text{on curve } \partial\Omega \text{ in } A), \tag{42}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{a}_3 \bigotimes \mathbf{a}_3$.

Obviously, Eqs. (33)–(34) [(38)–(39)] are the equilibrium equation and boundary condition for the bulk solid, whereas Eqs. (35)–(37) [(40)–(42)] can be regarded as the generalized Young-Laplace equations. It is seen that due to the existence of the surface/interface stress, the continuity of traction across an interface in classical continuum mechanics (e.g. [16]) is no longer valid.

In particular, if the surface/interface is isotropic relative to $\mathcal{K}_0$, and the constituent phases of the multi-phase body are also isotropic relative to $\mathcal{K}_*$, then γ can be expressed as a function of $J_1 = \mathrm{tr}\mathbf{V}_s$ and $J_2 = \det\mathbf{V}_s$, while ψ_0 can be expressed as a function of the following three invariants:

$$I_1 = \mathrm{tr}(\mathbf{C} \cdot \mathbf{B}^*),$$
$$I_2 = \mathrm{tr}(\mathbf{C}^{-1} \cdot (\mathbf{B}^*)^{-1}),$$
$$I_3 = (\det\mathbf{C})(\det\mathbf{B}^*),$$

where $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$, $\mathbf{B}^* = \mathbf{F}^* \cdot \mathbf{F}^{*T}$. Then, under the Eulerian description, the functional in Eq. (20) can be written as

$$\begin{aligned} \Pi(\mathbf{u}) \quad = \quad & \int_A \gamma(J_1, J_2)\, dA + \int_V \rho\, \psi(I_1, I_2, I_3)\, dv \\ & - \int_V \rho\, \mathbf{f} \cdot \mathbf{u}\, dv - \int_{\partial V_T} \bar{\mathbf{t}} \cdot \mathbf{u}\, ds. \end{aligned} \tag{43}$$

5 Determination of residual field

It is seen that the functional in Eq. (20) [or (43)] contains the residual field $\mathbf{F}^*$ or $\mathbf{B}^*$. It is determined by the intrinsic physical properties of the multi-phase material. The residual field in the reference configuration $\mathcal{K}_0$ is the solution corresponding to nil body force and boundary traction when the current configuration $\mathcal{K}$ coincides with the reference configuration $\mathcal{K}_0$. In this case, from Eqs. (40) and (41) and noting that $\boldsymbol{\sigma}_s^* = \gamma_0^* \mathbf{1}$, the stress field $\boldsymbol{\sigma}^*$ should satisfy

$$\boldsymbol{\sigma}^* \cdot \nabla = 0 \qquad (\text{in } V), \tag{44}$$

$$\boldsymbol{\sigma}^* \cdot \mathbf{N} = 0 \qquad (\text{on } \partial V), \tag{45}$$

$$\mathbf{A}_3 \cdot [\![\boldsymbol{\sigma}^*]\!] \cdot \mathbf{A}_3 = -\gamma_0^* \mathbf{1}_0 : \mathbf{b}_0 = -2\gamma_0^*\, H_0 \qquad (\text{on } A_0), \tag{46}$$

$$\mathbf{P}_0 \cdot [\![\boldsymbol{\sigma}^*]\!] \cdot \mathbf{A}_3 = -\nabla_{0s}\gamma_0^* \qquad (\text{on } A_0), \tag{47}$$

where $H_0 = \frac{1}{2}\mathbf{1_0} : \mathbf{b_0}$ is the mean curvature of the surface A_0. Another condition that is needed for the solution of the residual field is the coherence condition that maps two points, say, P_1 and P_2, in two surfaces into one, say, P, at the interface A_0. This condition may result in discontinuities of other stress components across A_0 than those related to the surface traction $[\![\boldsymbol{\sigma}^*]\!] \cdot \mathbf{A}_3$.

γ_0^* is the intrinsic physical attribute of the interface, which should be determined by experiments. When γ_0^* is a constant over the interface, $\nabla_{0s}\gamma_0^*$ is zero and hence $\mathbf{P}_0 \cdot [\![\boldsymbol{\sigma}^*]\!] \cdot \mathbf{A}_3 = 0$. The relation between $\boldsymbol{\sigma}^*$ and the residual deformation field can be obtained by setting $\mathbf{F} = \mathbf{I}$

in Eq. (19), i.e., $\boldsymbol{\sigma}^* = 2\rho_0 \mathbf{F}^* \cdot \left(\frac{\partial \psi_0}{\partial \tilde{\mathbf{C}}} \Big|_{\mathbf{F}=\mathbf{I}} \right) \cdot \mathbf{F}^{*T}$. Therefore, apart from a possible rigid body rotation, the deformation tensor $\mathbf{F}^*$ and the corresponding left Cauchy-Green tensors $\mathbf{B}^*$ can be determined from the given hyperelastic potentials for the constituent phases of a multi-phase body.

6 Illustrative examples

In this section, we will discuss the surface effect on the spherically symmetric deformation of compressible hyperelastic media. Investigations of the constitutive relations of compressible materials in bulk solids have received much attention in past decades by many researchers. For instance, the following six classes of hyperelastic potentials have been extensively studied by Ogden [17], Carroll [18, 19], Murphy [20, 21], Horgan [22, 23] and others. They are

$$\begin{array}{lll} \text{I} & \rho_0\psi = f(i_1) + c_2\,(i_2-3) + c_3\,(i_3-1)\,, & f''(i_1)\neq 0 \\ \text{II} & \rho_0\psi = c_1\,(i_1-3) + g(i_2) + c_3\,(i_3-1)\,, & g''(i_2)\neq 0 \\ \text{III} & \rho_0\psi = c_1\,(i_1-3) + c_2\,(i_2-3) + h(i_3), & h''(i_3)\neq 0 \\ \text{IV} & \rho_0\psi = c_1 i_1 i_2 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, & c_1\neq 0 \\ \text{V} & \rho_0\psi = c_1 i_2 i_3 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, & c_1\neq 0 \\ \text{VI} & \rho_0\psi = c_1 i_1 i_3 + c_2 i_1 + c_3 i_2 + c_4 i_3 + c_5, & c_1\neq 0 \end{array} \tag{48}$$

where $f,\ g$ and h are material functions of their indicated arguments, c_1, c_2, c_3, c_4, c_5 are material constants, which differ from class to class, and i_1, i_2 and i_3 are the principal invariants of the stretch tensor:

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3\ , \quad i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\ , \quad i_3 = \lambda_1\lambda_2\lambda_3 \tag{49}$$

It should be noted that in this paper, the above stretch tensor should be understood as $\tilde{\mathbf{U}} = \tilde{\mathbf{C}}^{\frac{1}{2}}$, which is evaluated from the fictitious stress-free configuration $\mathcal{K}_*$ to the current configuration $\mathcal{K}$. Materials of class I are called the harmonic materials introduced by John [24], and materials of class III are generalized Varga materials introduced by Horgan [22].

Restrictions imposed on the hyperelastic potential are usually required. For example, these restrictions are: i) In the stress-free configuration, the hyperelastic potential and the stress field vanish; ii) In the case of infinitesimal deformations, the hyperelastic potential function should reduce to the classical one of the linear theory. According to the above restrictions, for class I, we obtain

$$f(3) = 0, \quad f'(3) = -2c_2 - c_3 \tag{50}$$

$$c_2 + c_3 = -2\mu < 0 \tag{51}$$

$$f''(3) = K + \frac{4}{3}\mu \tag{52}$$

where $\mu(>0)$ and $K(>0)$ are material constants. For class III, we have

$$h(1) = 0 \quad , \quad h'(1) = -c_1 - 2c_2 \tag{53}$$

$$c_1 + c_2 = 2\mu > 0 \tag{54}$$

$$h''(1) = K + \frac{4}{3}\mu \tag{55}$$

where $\mu(>0)$ and $K(>0)$ are also material constants, which differ from those in Eqs. (51) and (52).

Now consider a radially symmetric deformation described by

$$r = r(R), \quad \theta = \Theta, \quad \phi = \Phi \tag{56}$$

where (R,Θ,Φ) and (r,θ,φ) are the spherical polar coordinates of a particle before and after deformation, with the principal stretches to be

$$\lambda_1 = \frac{dr(R)}{dR} = r', \quad \lambda_2 = \lambda_3 = \frac{r(R)}{R} \tag{57}$$

The radial and hoop components of the Cauchy stress $\boldsymbol{\sigma}$ can be expressed in terms of hyperelastic potential $\rho_0\psi$ as follows

$$\sigma_r = \rho_0 \left(\frac{1}{\lambda_2^2}\right) \frac{\partial \psi_0}{\partial \lambda_1}, \quad \sigma_\theta = \sigma_\phi = \rho_0 \left(\frac{1}{\lambda_1 \lambda_2}\right) \frac{\partial \psi_0}{\partial \lambda_2} \tag{58}$$

Hence, the equilibrium equation in the absence of body force can be written as

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R^2 \frac{\partial \psi_0}{\partial \lambda_1}\right] - 2R\frac{\partial \psi_0}{\partial \lambda_2} = 0 \tag{59}$$

For class I, the solution of Eq. (59) is

$$r(R) = k_1 R + \frac{k_2}{R^2} \tag{60}$$

which implies $i_1 = 3k_1$ is a constant. For class III, the solution of Eq. (59) gives

$$r(R) = \left(k_3 R^3 + k_4\right)^{\frac{1}{3}} \tag{61}$$

which implies $i_3 = k_3$ is a constant. This result was also obtained by Haughton [25] for $c_2 = 0$ in Eq. (48).

In order to study the surface effect, let us consider a homogeneous hyperelastic medium of infinite extent first. Within this hyperelastic medium, there is a spherical region D with the radius R_*. The medium is not subjected to any external loading, and it is stress-free, which corresponds to the configuration $\mathcal{K}_*$. Now imagine that the spherical region D is removed from this infinite medium. Then this infinite medium will contain a cavity, and the region D will become an isolated particle. Since both boundaries of the cavity and the particle are newly created surfaces, which give rise to the surface energies and the surface stresses, there exist the elastic fields both in the infinite medium containing the cavity and in the particle. These elastic fields are induced by the surface energies and the surface stresses even there is no external loading, and we refer to these (deformed) self-equilibrium states as reference configurations both for the infinite medium containing the cavity and for the particle.

Problem 1: Suppose that the hyperelastic potential of the above mentioned medium can be described by that of the harmonic material, and the surface is isotropic relative to the reference configuration, so, the deformation is spherically symmetric with respect to the centre of the cavity. The problem is how to determine the radius $R_0 = r(R_*)$ of this newly formed spherical cavity.

Noting that the deformation is given by Eq. (60), and λ_1 and λ_2 tend to 1 at the infinity, we obtain $k_1 = 1$ and $i_1 = 3$. Hence σ_r in Eq. (58) can be written as

$$\sigma_r = \left(\frac{1}{\lambda_2}\right)^2 \left[f'(3) + 2c_2\lambda_2 + c_3\lambda_2^2\right] = \left(\frac{1}{\lambda_2}\right)^2 \left[2c_2\left(\lambda_2 - 1\right) + c_3\left(\lambda_2^2 - 1\right)\right]$$

By using the generalized Young-Laplace equation on $r = r(R_*)$ at the reference configuration

$$\sigma_r|_{r(R_*)} = \frac{2\gamma_0^*}{r(R_*)} = \left(\frac{1}{\lambda_*}\right)\frac{2\gamma_0^*}{R_*}$$

we have

$$c_3\lambda_*^2 + 2\left(c_2 - \frac{\gamma_0^*}{R_*}\right)\lambda_* - (2c_2 + c_3) = 0 \tag{62}$$

where $\lambda_* = \frac{r(R_*)}{R_*}$ and γ_0^* is the residual surface stress.

Solutions of Eq. (62) are given by

$$\lambda_* = \begin{cases} -\left(\frac{c_2}{c_3} - \frac{\gamma_0^*}{c_3 R_*}\right) + \sqrt{\left(\frac{c_2}{c_3} - \frac{\gamma_0^*}{c_3 R_*}\right)^2 + \left(1 + \frac{2c_2}{c_3}\right)} & (if\ c_3 < 0) \\ \frac{1}{1 - \frac{\gamma_0^*}{c_2 R_*}} & (if\ c_3 = 0) \\ -\left(\frac{c_2}{c_3} - \frac{\gamma_0^*}{c_3 R_*}\right) - \sqrt{\left(\frac{c_2}{c_3} - \frac{\gamma_0^*}{c_3 R_*}\right)^2 + \left(1 + \frac{2c_2}{c_3}\right)} & (if\ c_3 > 0) \end{cases} \tag{63}$$

In view of Eq. (51), $c_3 < 0$, $c_3 = 0$ and $c_3 > 0$ correspond to $\left(1 + \frac{c_2}{c_3}\right) > 0$, $c_2 < 0$ and $\left(1 + \frac{c_2}{c_3}\right) < 0$, respectively, which implies that λ_* in Eq. (63) is always less than 1 if γ_0^* is positive. This means that the radius of the cavity satisfies $R_0 = r(R_*) = \lambda_* R_* < 1$, due to the surface energy effect.

Problem 2: Suppose that the above-mentioned hyperelastic medium is made of the harmonic material given by Eq. (48) and following Ogden [17], the material function $f(i_1)$ is assumed to be

$$f(i_1) = (2c_2 + c_3)\left(1 - \frac{i_1^3}{27}\right) \tag{64}$$

Moreover, suppose that the newly-created surface of the particle is isotropic relative to the reference configuration. Thus, due to the spherical symmetry of the problem, the deformed particle is also spherical. The problem is how to determine the radius $R_0 = r(R_*)$ of this particle.

Obviously, the restriction of Eq. (50) is automatically satisfied by Eq. (64). From $f''(3) = -\frac{2}{3}(2c_2 + c_3)$ and Eqs. (51) and (52), we have $c_2 = -\frac{3}{2}\,K(< 0)$, and $c_3 = -2\mu + \frac{3}{2}K$. The deformation of the particle is also described by Eq. (60). At the centre of the particle, the requirement $r(0) = 0$ leads to $k_2 = 0$, so Eq. (60) becomes $r(R) = k_1 R$, which gives $\lambda_2 = k_1$, $i_1 = 3k_1$. Hence the radial components of the Cauchy stress in Eq. (58) can be expressed by

$$\sigma_r = \left(\frac{1}{k_1}\right)^2 \left(f'(3k_1) + 2c_2 k_1 + c_3 k_1^2\right)$$

In view of Eq. (64), $f'(3k_1)$ in the above equation can be written as $f'(3k_1) = -(2c_2 + c_3)\,k_1^2$.

On the surface of the particle, the generalized Young-Laplace equation corresponding to the reference configuration is

$$-\sigma_r|_{r(R_*)} = \frac{2\gamma_0^*}{r(R_*)} = \frac{2\gamma_0^*}{k_1 R_*}$$

which gives $\left[c_2 k_1 - \left(c_2 + \frac{\gamma_0^*}{R_*}\right)\right] k_1 = 0$. Thus from $\lambda_2 = k_1 > 0$, we obtain

$$k_1 = 1 + \frac{\gamma_0^*}{c_2 R_*} = 1 - \frac{2\gamma_0^*}{3K R_*} \tag{65}$$

This means that at the equilibrium state, the radius of the particle $r(R_*) = k_1 R_*$ is always less than R_* as long as γ_0^* is positive.

From the above problems, it can be seen that the radius of either the cavity or the particle will not be the same as that of the spherical region D, if there is the surface effect. Conversely, for an infinite matrix material containing a cavity or for a particle, if there exists the surface effect, there must be an elastic field induced by the surface energy and the surface stress even there is no external loading. This elastic field can be described by the deformation from the fictitious stress-free configuration $\mathcal{K}_*$ to the reference configuration $\mathcal{K}$.

Next, consider a hollow sphere which is subjected to hydrostatic loading. This hollow sphere can be used as the "composite spherical model" to predict the effective radial stress-deformation response of a porous material under finite deformation. For example, the spherically symmetric deformations of porous materials have been studied by Hashin [26] for an incompressible hyperelastic matrix of the Mooney-Rivlin type, and by Horgan [22], and Aboudi and Arnold [27] for compressible, nonlinear elastic matrix materials. In order to study the surface effect on the effective radial stress-deformation response of the porous material, the following problem should be solved. However, it should be noted, that in the "composite spherical model", the outer surface of the hollow sphere is not a true material surface, but an imaginary one, so there is no surface stress on the outer boundary of this hollow sphere.

Problem 3: Suppose that the hollow sphere is made of the generalized Varga material. The inner and outer radii of the hollow sphere are B_0 and D_0, respectively, in the reference configuration $\mathcal{K}_0$ and the inner surface is isotropic relative to $\mathcal{K}_0$. Moreover, the surface energy γ is assumed to be a linear function of J_1 and J_2:

$$\gamma = \gamma_0 + \gamma_1(J_1 - 2) + \gamma_2(J_2 - 1) \tag{66}$$

where $\gamma_0,\ \gamma_1$ and γ_2 are material parameters of the surface, which represent the intrinsic physical properties of the surface. The problem to be solved is: when the outer boundary of this hollow sphere is subjected to a radial traction σ at the current configuration, how to determine the radial stress-deformation response of this hollow sphere?

The above problem can be solved by the following two steps: a) In the reference configuration $\mathcal{K}_0$, the outer boundary of the hollow sphere is stress-free. However, due to the surface effect at the inner surface, there must be an elastic field induced by the surface stress. According to Eq. (61), this elastic field can be described by the deformation from a fictitious stress-free configuration $\mathcal{K}_*$ to the reference configuration $\mathcal{K}_0$ as follows:

$$r(R_*) = \left(k_3 R_*^3 + k_4\right)^{\frac{1}{3}} \tag{67}$$

or

$$R_*(r) = \left(\frac{1}{k_3} r^3 - \frac{k_4}{k_3}\right)^{\frac{1}{3}} \tag{68}$$

Hence, in the fictitious stress-free configuration $\mathcal{K}_*$, the inner and outer radii of this hollow sphere should be

$$B_* = \left(\frac{1}{k_3} B_0^3 - \frac{k_4}{k_3}\right)^{\frac{1}{3}}, \quad \text{and} \quad \mathrm{D}_* = \left(\frac{1}{\mathrm{k}_3} \mathrm{D}_0^3 - \frac{\mathrm{k}_4}{\mathrm{k}_3}\right)^{\frac{1}{3}} \tag{69}$$

So, the first step is to determine k_3 and k_4. b) When a radial traction σ is applied to the outer boundary of the hollow sphere in the current configuration $\mathcal{K}$, the inner and outer radii of this hollow sphere will become B and D, respectively. The deformation from the fictitious configuration $\mathcal{K}_*$ to the current configuration $\mathcal{K}$ can be described by

$$r(R_*) = \left(k_5 R_*^3 + k_6\right)^{\frac{1}{3}} \tag{70}$$

with $B = \left(k_5 B_*^3 + k_6\right)^{\frac{1}{3}}$ and $D = \left(k_5 D_*^3 + k_6\right)^{\frac{1}{3}}$. So the second step is to determine k_5 and k_6 .

In the real problem, we are only interested in the deformation process from the reference configuration $\mathcal{K}_0$ to the current configuration $\mathcal{K}$, so the effective stretch may be defined by

$$\bar{\lambda} = \frac{D}{D_0} = \left(\frac{k_5 D_*^3 + k_6}{k_3 D_*^3 + k_4}\right)^{\frac{1}{3}} \tag{71}$$

where D_* is given by Eq. (69).

If for every given radial traction σ, the effective stretch $\bar{\lambda}$ can be calculated, then the effective radial stress-deformation response, i.e., the relationship between σ and $\bar{\lambda}$, can be obtained.

In step a), the radial component of the Cauchy stress in Eq. (58) can be written as

$$\sigma_r = \left(\frac{1}{\lambda_2}\right)^2 \left[c_1 + 2c_2\lambda_2 + h'(i_3)\lambda_2^2\right] \tag{72}$$

where

$$\lambda_2 = \left(\frac{k_3 r^3(R_*)}{r^3(R_*) - k_4}\right)^{\frac{1}{3}}$$

and $i_3 = k_3$.

In the reference configuration, the outer boundary is traction-free, so we have

$$c_1 + 2c_2 \left(\frac{k_3 D_0^3}{D_0^3 - k_4} \right)^{\frac{1}{3}} + h'(k_3) \left(\frac{k_3 D_0^3}{D_0^3 - k_4} \right)^{\frac{2}{3}} = 0 \tag{73}$$

The Young-Laplace equation at the inner surface is given by

$$c_1 \left(\frac{B_0^3 - k_4}{k_3 B_0^3} \right)^{\frac{2}{3}} + 2c_2 \left(\frac{B_0^3 - k_4}{k_3 B_0^3} \right)^{\frac{1}{3}} + h'(k_3) = \frac{2\gamma_0^*}{B_0} \tag{74}$$

Hence k_3 and k_4 can be determined from Eqs. (73) and (74). Especially, if there is no surface stress: $\gamma_0^* = 0$, we have $k_3 = 1,\ k_4 = 0$, and the fictitious configuration coincides with the reference configuration.

In step b), the radial component of the Cauchy stress is still given by Eq. (72), but λ_2 and i_3 should be replaced by $\lambda_2 = \frac{(k_5 R_*^3 + k_6)^{\frac{1}{3}}}{R_*}$ and $i_3 = k_5$. In the current configuration, the radial traction on the outer boundary can be written as

$$\frac{D_*^2}{(k_5 D_*^3 + k_6)^{\frac{2}{3}}} c_1 + \frac{2D_*}{(k_5 D_*^3 + k_6)^{\frac{1}{3}}} c_2 + h'(k_5) = \sigma \tag{75}$$

and the generalized Young-Laplace equation at the inner surface is

$$\frac{B_*^2}{(k_5 B_*^3 + k_6)^{\frac{2}{3}}} c_1 + \frac{2B_*}{(k_5 B_*^3 + k_6)^{\frac{1}{3}}} c_2 + h'(k_5) = \frac{2\sigma_s}{B_0 \lambda_s} \tag{76}$$

where σ_s is the surface stress, and from Eq. (15) and Eq. (66), it can be expressed as

$$\sigma_s = \gamma_0 + \gamma_1 (3\lambda_s - 2) + \gamma_2 (2\lambda_s^2 - 1) \tag{77}$$

and

$$\lambda_s = \frac{B}{B_0} = \left(\frac{k_5 B_*^3 + k_6}{k_3 B_*^3 + k_4} \right)^{\frac{1}{3}} \tag{78}$$

Hence k_5 and k_6 can be determined from Eqs. (75) and (76), as long as k_3 and k_4 are known.

The spherically symmetric deformations of other classes of hyperelastic materials have also been studied in the literature, e.g. Murphy [20, 21], Chung et al. [28], Hill [29], and Murphy and Biwa [30], where no surface effect was considered. The treatment in this section can also be employed to study the surface effect on the spherically symmetric deformations of other classes of compressible hyperelastic media.

7 Concluding remarks

The effect of the surface/interface energy becomes important in the physical and mechanical properties of materials with a characteristic size at the nanoscale. In order to systematically study this effect, the governing equations that are based upon the physical mechanisms are needed. For multi-phase hyperelastic media, in terms of the physical origin of the surface/interface energy and stress, there should be a residual elastic field due to the existence of the surface/interface, even under no external loading. The deformation of the body under external loading will depend upon this residual elastic field. Therefore, the concept of a fictitious stress-free configuration is first introduced in this paper to facilitate the study of the complete deformation of the body under external loading. Then, this paper presents a new energy functional for multi-phase hyperelastic solids which takes into account the contribution of the interface energy. All the needed field equations and boundary conditions, including the generalized Young-Laplace equation for a multi-phase solid, for solving the elastostatic problem are obtained from the stationary condition of the functional. Expressions for the interface stress in terms of the interface free energy are given and discussed. To demonstrate the necessity of the fictitious stress-free configuration and the applicability of the theoretical framework, some basic problems in finite deformation, but with the surface/interface energy effect, are solved. The effect of the surface/interface energy on the deformation of the hyperelastic solids is manifested. The present analysis provides not only a framework for studying elastostatic problems of hyperelastic bodies that involve the surface/interface effect at finite deformation, but also a basis for studying the effective mechanical properties of multi-phase materials with the surface/interface stress effect. Related work in the latter subject will be reported elsewhere.

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