Resistance Curve

Resistance curve (a commonly used phrase). In the previous lectures, we have taken the fracture energy of a material to be a constant. The constant fracture energy is an idealization. The fracture energy can be a function of the extension of the crack:

$$\Gamma = \Gamma\left(a - a_o\right),$$

where a_o is the initial length of the crack, a is the current length of the crack, and $a - a_o$ is the extension of the crack. This function is known as the resistance curve (*R*-curve).

Any specific fracture process will lead to a *R*-curve. Consider, for example, the gradual development of the inelastic layer as the crack advances. Here is a frame-by-frame movie.

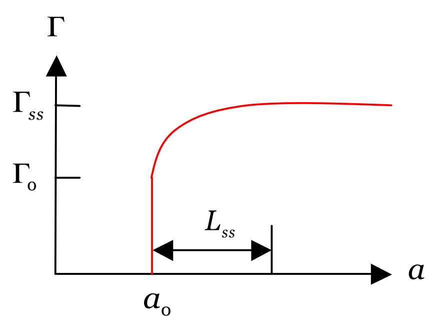
Frame 1. A plate of a steel is cut by a diamond saw. The initial length of the crack, a_o, is *not* a material property. No force is applied to the plate yet.

Frame 2. When a load is applied, under the small-scale yielding condition, the applied load can be converted to the energy release rate by solving a boundary-value problem of Type A. That is, the applied load can be represented by the energy release rate *G*.

Frame 3. Apply a small load. A small region of the material around the tip of the crack yields, but the length of the crack remains to be a_o.

Frame 4. Increase the load still more. Above a critical force, $G = \Gamma_o$, the crack starts to advance. The larger the load, the longer the crack. Denote the increase in the length of the crack by Δa.

Frame 5. When the crack advances in a **steady state**, the thickness of the inelastic layer remains constant as the crack advances. The load *G* attains a plateau (a steady-state), Γ_{ss}.



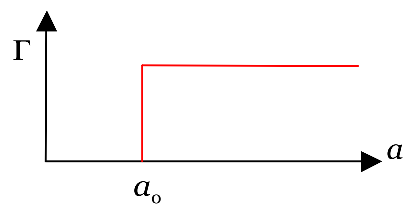
Under the small-scale yielding condition, the *R*-curve is a material property. The *R*-curve is independent of the initial length of the crack, a_o. Nor does the *R*-curve depend on the geometry of the specimen. Parameters of importance include

- the initiation fracture energy, Γ_o,

- the steady-state fracture energy, Γ_{ss}, and
- the increment of the length of the crack needed to attain the steady state, L_{ss}.

All these parameters are material properties. The ratio Γ_{ss}/Γ_0 ranges from a factor of 2 to a factor of thousands. The length L_{ss} is comparable to the size of the plastic zone when the crack advances in the steady state.

To satisfy the small scale yielding condition, L_{ss} must be small compared to the size of the specimen (i.e., the length of the crack, the length of the ligament, the thickness of the specimen, etc.) For silica, $L_{ss} \sim 1\text{nm}$. Thus, for practical purpose, the fracture energy is a constant, independent of the extension of the crack, and the *R*-curve looks like a step function.



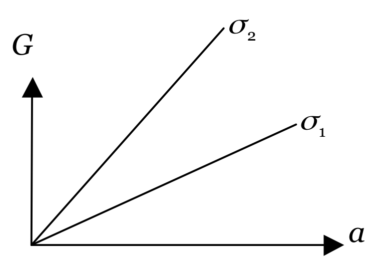
For a ductile steel, $L_{ss} \sim 1\text{cm}$, the fracture energy will increase from Γ_0 to Γ_{ss} after the crack extends for a length comparable to L_{ss}.

The *R*-curve can also be represented by a function $K_R(\Delta a)$.

Loading curve (not a commonly used phrase). For a Griffith crack, the energy release rate is

$$G = \pi \frac{\sigma^2 a}{\bar{E}}.$$

Consider an experiment in which the applied stress σ remains constant when the crack extends. In such an experiment, the energy release rate is linear in the length of the crack, a. Plot the energy release rate as a function of the length of the crack for several levels of the applied stress.



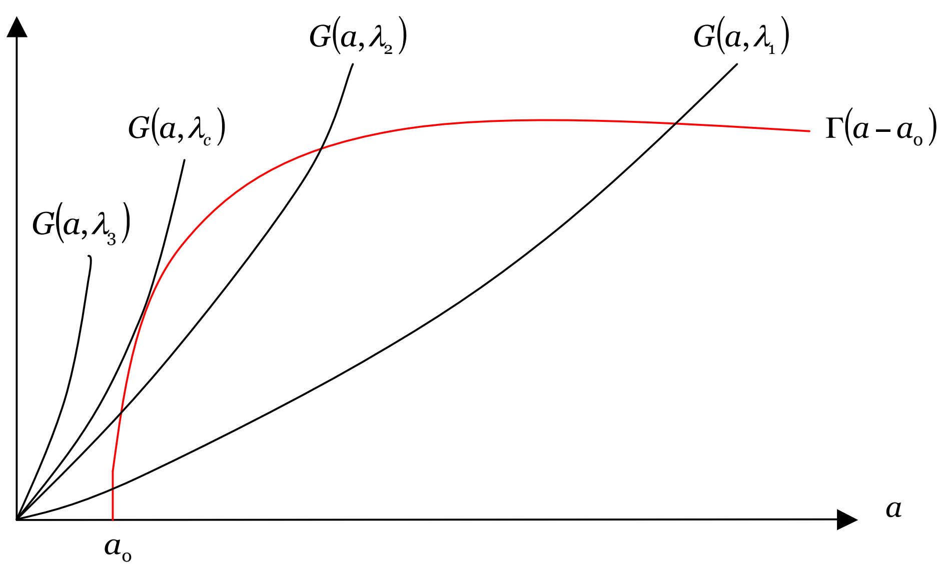
Let λ be a loading parameter, which can be an applied force, or a prescribed displacement. The energy release rate depends on both the loading parameter and the length of the crack, namely,

$$G = G(a,\lambda).$$

This function of two independent variables represents a family of curves, which I will call the loading curves. The loading curves can also be represented by the function $K(a,\lambda)$.

Stable extension of a crack. For a crack in a body under a load, will the crack remain stationary, or grow then stop (stable), or grow across the entire body (unstable)? Under the small-scale yielding condition, the answer depends on three inputs:

- The resistance curve of the material, $\Gamma(a-a_0)$.
- The loading curves of the specimen, $G(a,\lambda)$.
- The initial length of the crack, a_0.



The stability of a crack depends on how the two curves intersect. When the loading parameter is fixed at λ_1, the *R*-curve is above the *L*-curve for all cracks longer than a_0, and the crack will not grow. When the loading parameter is fixed at λ_2, the crack will grow and then stop. When the loading parameter is fixed at λ_3, the crack will grow and then stop. A critical load λ_c exists, below which the crack will stop. This critical condition satisfies two simultaneous equations

$$G(a,\lambda) = \Gamma(a-a_0),$$
$$\frac{\partial G(a,\lambda)}{\partial a} = \frac{d\Gamma(a-a_0)}{da}.$$

These two equations determine two variables: the critical load and the associated length of the crack.

Loading curves of various trends. In the above discussion, loading curves of a particular trend are assumed. The loading curve depends on the boundary conditions of the specimen. It will be helpful to look at several examples.

Example 1: Griffith crack, constant fracture energy. Draw three figures. Draw the *L*-curves for several fixed levels of stress. Draw the *R*-curve, $\Gamma(\Delta a)$. Give the initial crack size a_o and draw the loading curves and the *R*-curve on the third figure.

Example 2: Griffith crack, rising *R*-curve. Discuss the experimental determination of the *R*-curve using the Griffith crack.

Example 3: Double cantilever beam, force control. We derived the energy release rate using the elementary beam theory. The result is

$$G = \frac{12}{EH^3}\left(\frac{Pa}{B}\right)^2.$$

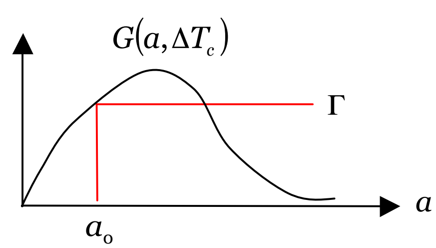
Under a fixed force P, the loading curve is an increasing function. For now assume that the crack resistance Γ is a constant independent of the crack length. The crack is unstable. For a given initial crack size, the crack is stationary when the stress is below a critical value, and extends rapidly when the stress exceeds the critical value.

Example 4: Double cantilever beam, displacement control. Under displacement control (e.g., by inserting a wedge), the energy release rate is given by

$$G = \frac{3EH^3\Delta^2}{16a^4}.$$

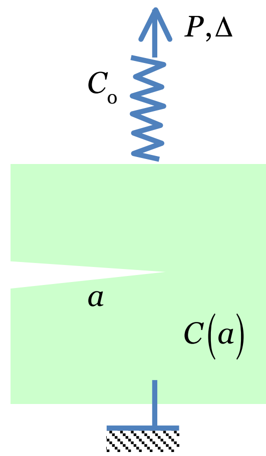
The energy release rate *decreases* when the crack length increases.

Example 5: A crack in a matrix and near a fiber. In this case, the energy release rate is a function of the crack length and the drop in temperature, $G(a,\Delta T)$. At a fixed temperature, the energy release rate as a function of the length of the crack is not monotonic. The energy release rate reaches the maximum when $a = R/2$. For simplicity, assume a constant fracture energy. Distinguish two cases: $a_o < R/2$ and $a_o > R/2$.



A cracked body in series with a compliant spring. Orowan (1955) pictured a long wire containing a crack. Describe this picture. Consider a cracked body in series with a spring. The compliance of the body C depends on the length of the crack, but the compliance of the spring C_o is independent of the length of the crack. When a loading machine prescribes a displacement Δ at the end of the spring, the force is P. The displacement relates to the force by

$$\Delta = \left(C + C_o\right)P$$



The elastic energy in the body and the spring is

$$U = \frac{1}{2}P\Delta .$$

When the crack extends, the displacement is held constant, so that the energy release rate is

$$G = -\frac{\partial U\left(\Delta, a\right)}{B\partial a} .$$

where B is the thickness of the specimen.

A combination of the above expressions gives the energy release rate

$$G = \frac{\Delta^2}{2B\left(C + C_o\right)^2}\frac{dC}{da} .$$

The slopes of the loading curves are

$$\frac{\partial G\left(\Delta, a\right)}{\partial a} = \frac{\Delta^2}{2B\left(C + C_o\right)^2}\frac{d^2C}{da^2} - \frac{\Delta^2}{4B\left(C + C_o\right)^3}\left(\frac{dC}{da}\right)^2 .$$

Use the expression of the energy release rate to replace the displacement, and we obtain that

$$\frac{\partial G\left(\Delta, a\right)}{\partial a} = G\left[\frac{d^2C}{da^2}\left(\frac{dC}{da}\right)^{-1} - \frac{1}{2\left(C + C_o\right)}\frac{dC}{da}\right]$$

Let a_o be the initial length of the crack. When the extension of the crack initiates, $G = \Gamma\left(0\right)$. The initial extension is unstable when

$$\frac{\partial G\left(\Delta, a\right)}{\partial a} > \frac{d\Gamma\left(a - a_o\right)}{da}$$

When the spring is more compliant, the slope of the loading curve is steeper. Thus, a compliant spring may destabilize the extension of a crack.

Historical Notes

Orowan discussed fracture resistance as a function of the extension of the crack. He also discussed the stability of the crack.

- E. Orowan, Condition of high-velocity ductile fracture. Journal of Applied Physics 26, 900-902 (1955).

Further discussions of stability of the crack is in the following paper

- J.W. Hutchinson, P.C. Paris, "Stability Analysis of J-Controlled Crack Growth in Elastic-Plastic Fracture," in ASTM STP 668, edited by J. D. Landes et al., American Society for Testing and Materials, (1979). http://www.seas.harvard.edu/hutchinson/papers/352.pdf