7.8 RIGID LINKS. RIGID ELEMENTS

Rigid members impose relationships among d.o.f. This circumstance is sometimes called a multipoint constraint.[3] In the present section we consider rigid members as an application of coordinate transformation.

Rigid Links. Imagine that a plate is to be reinforced by a beam (Fig. 7.8-1). Nodes of the beam do not coincide with nodes of the plate. (If nodes were coincident, the beam–plate connection would be easy; one would simply assemble elements in the usual way.) Even with an offset beam, it is still possible to connect beam and plate in such a way that d.o.f. of only the plate appear in the assembled structure. The procedure for doing so is now described.

The procedure invokes a transformation that makes beam d.o.f. at nodes 3 and 4 "slave" to "master" d.o.f. at nodes 1 and 2 in the plate. This is accomplished by adding imaginary, weightless, rigid links—one between nodes 1 and 3 and another between nodes 2 and 4. We assume that the beam has bending stiffness (associated with d.o.f. w_3, θ_3, w_4, and θ_4) and axial stiffness (associated with d.o.f. u_3 and u_4). These six d.o.f. must be incorporated in the transformation relation. At the left end, the transformation is

$$\begin{Bmatrix} u_3 \\ w_3 \\ \theta_3 \end{Bmatrix} = [\mathbf{T}_\ell] \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \end{Bmatrix}, \qquad \text{where} \quad [\mathbf{T}_\ell] = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.8-1}$$

A similar transformation is written at the right end by replacing subscripts 1 by 2 and 3 by 4. We see that d.o.f. u_3 and u_4 are activated by θ_1 and θ_2. Thus, because of the rigid links, axial stiffness of the beam is seen as bending stiffness by the plate d.o.f.

Let {r′} and [k′] be beam element matrices associated with d.o.f. at nodes 3 and 4 (see Fig. 7.5-2*a* for [k′]). Transformed arrays {r} and [k], associated with d.o.f. at plate nodes 1 and 2, are

$$\begin{aligned} \{\mathbf{r}\} &= [\mathbf{T}]^T\{\mathbf{r}'\} \\ [\mathbf{k}] &= [\mathbf{T}]^T[\mathbf{k}'][\mathbf{T}] \end{aligned} \qquad \text{where} \quad \underset{6\times 6}{[\mathbf{T}]} = \begin{bmatrix} \mathbf{T}_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_\ell \end{bmatrix} \tag{7.8-2}$$

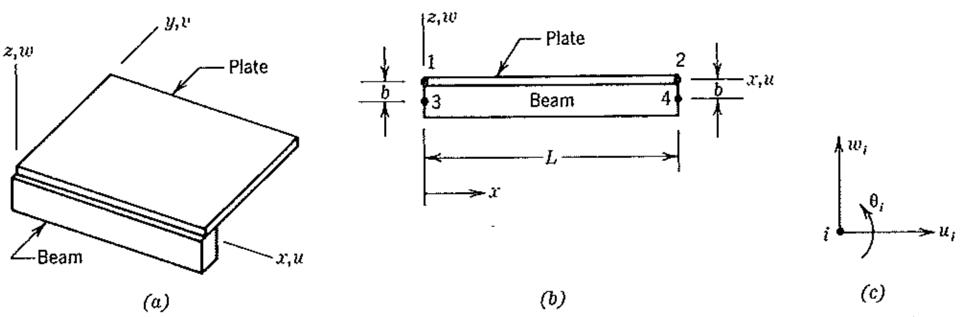

Figure 7.8-1. (*a*) A reinforcing beam joined to one edge of a plate element. (*b*) Side view. (*c*) Typical node *i* (*i* = 1, 2, 3, 4), showing d.o.f. considered in the coordinate transformation.

[3] Constraints are discussed in detail in Chapter 9.

Clearly this procedure can be extended to deal with a stiffener that is arbitrarily oriented in space, and with rigid links that are not perpendicular to the stiffener.

The foregoing transformation introduces an error that can cause displacements to be significantly overestimated [7.1]. The error can be attributed to incomplete coupling between beam and plate. Axial displacement in the beam should be

$$u_{\text{beam}} = u_{\text{plate}} + b\theta_{\text{plate}} \tag{7.8-3}$$

Imagine, for example, that all d.o.f. of the plate in Fig. 7.8-1*b* are zero but w_2. Then w_{plate} is cubic in x and θ_{plate} is quadratic in x. Hence, according to Eq. 7.8-3, u_{beam} should be quadratic in x. However, Eq. 7.8-1 yields $u_3 = u_4 = 0$; hence, $u_{\text{beam}} = 0$. Thus, for this deformation mode, beam and plate bending stiffnesses are simply added rather than being combined in a way that recognizes a common neutral axis. For a test case in which a uniform cantilever was loaded by a transverse tip force, with n plate elements along the length, tip displacement was overestimated by 69% for $n = 1$, 17% for $n = 2$, and 4.3% for $n = 4$ [7.1]. The error tends toward zero as each element approaches a state of constant curvature.

A method that eliminates the error was suggested by Miller [7.2]. He introduces axial displacement d.o.f. at $x = L/2$, say u_5 in the plate and u_6 in the beam. Axial displacement in the beam is now quadratic in x, as is desired. The axial stiffness portion of [k′] is 3 by 3 and is associated with u_3, u_4, and u_6 (see Section 6.2). The transformation is essentially that of Eq. 7.8-2, augmented by

$$u_6 = u_5 + b\left(\frac{dw}{dx}\right)_{x=L/2} \tag{7.8-4}$$

where plate rotation dw/dx depends on w_1, θ_1, w_2, and θ_2. Transformation causes the 7 by 7 beam element stiffness matrix to operate on d.o.f. u_1, w_1, θ_1, u_2, w_2, θ_2, and u_5. This matrix is then combined with the plate element stiffness matrix (whose row and column corresponding to u_5 are null). Finally, condensation removes u_5, thus producing a combined [k] that operates on the usual plate element d.o.f.

Another difficulty, encountered in dynamic problems, is that the transformation converts a diagonal beam mass matrix [m′] to a nondiagonal mass matrix [m]. Ad hoc adjustments of [m] can make it diagonal again.

Rigid Elements. A rigid element might be used to model part of a linkage mechanism that couples elastic bodies. Or a particular element might be of much higher modulus than surrounding elements. In the latter case, errors of the type discussed in Section 18.2 are likely, and it is better to make the element perfectly rigid rather than very stiff.

Imagine that the triangle of Fig. 7.8-2 is to be idealized as perfectly rigid.

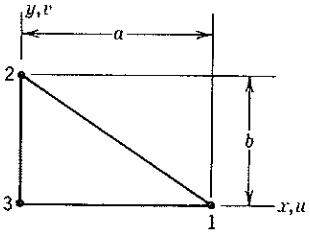

Figure 7.8-2. A plane triangle. Other elements of the structure are connected to it but are not shown.

Therefore, its motion is completely described by three d.o.f., say u_1, v_1, and u_2. These d.o.f. are related to the original six d.o.f. by the transformation

$$\{\mathbf{d}'\} = [\mathbf{T}]\{\mathbf{d}\} \qquad \text{or} \qquad \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a/b & 1 & a/b \\ 1 & 0 & 0 \\ -a/b & 1 & a/b \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \end{Bmatrix} \tag{7.8-5}$$

in which $u_3 = u_1$ and $v_2 = v_3 = v_1 - \theta a$, where $\theta = (u_1 - u_2)/b$ is a small rigid-body rotation. Transformation according to Eq. 7.8-5 is applied to all elements of the structure that contain any of the d.o.f. v_2, u_3, and v_3. Thus, v_2, u_3, and v_3 no longer appear as d.o.f. in {**D**}. The particular stiffness coefficients of triangle 1–2–3 do not matter; they are overridden by the rigid-body constraint.

The choice $\{\mathbf{d}\} = \lfloor u_1 \quad v_1 \quad u_2 \rfloor^T$ is not unique, and would be unacceptable if node numbers were rearranged so that $y_2 - y_1 = b = 0$. Not only would there be a division by zero in Eq. 7.8-5, but the use of u_1 and u_2 as independent d.o.f. would contradict the assumption that the triangle is rigid.

PROBLEMS

Section 7.2

7.1 If a vector **V** has length L, then $\mathbf{V}\cdot\mathbf{V} = L^2$ regardless of the coordinate system in which **V** resides. Hence, using Eq. 7.2-1, show that $\Sigma\, \ell_i = 1$, $\Sigma\, \ell_i m_i = 0$, and so on (six such relations altogether).

7.2 (a) Let $x' = -x$ and $y' = -y$. What is [Λ] in Eq. 7.2-1 if both coordinate systems are right-handed?
(b) Similarly, what is [Λ] if $x' = y$ and $z' = z$?

7.3 (a) If $z = z'$ and x' is located at a counterclockwise angle θ from x, what is [Λ] in Eq. 7.2-1?
(b) For this [Λ], show that $[\Lambda]^{-1} = [\Lambda]^T$.

Section 7.3

7.4 Let [**E**′] be 3 by 3, as for a plane stress problem. Show that Eq. 7.3-10 yields [**E**′] = [**E**] if the material is isotropic.

7.5 Let an orthotropic material have principal directions x', y', and z (i.e., axes z' and z coincide). Write the 6 by 6 matrix $[\mathbf{T}_\epsilon]$ for this situation. Express your answer in terms of sin β and cos β.

7.6 Consider a plane problem for which the 3 by 3 matrix [**E**′] is diagonal, with $E'_{11} = E_a$, $E'_{22} = E_b$, and $E'_{33} = G$. What is [**E**] for an arbitrary angle β in Fig. 7.3-1? As a partial check on your answer, try the case $\beta = \pi/2$.

7.7 Is $[\mathbf{T}_\epsilon]$ of Eq. 7.3-11 an orthogonal matrix?