Saint-Venant's Principe of the Problem of the Cylinder

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Abstract

The problem of the infinite axisymmetrical circular cylinder loaded by an equilibrium system of forces on its near end is discussed and its formulation of Special Saint-Venant's Principle is established. It is essential to develop mathematical theories of Special Saint-Venant's Principle one by one if Elasticity has to be constructed to be rational, logical, rigorous and secure mechanics.

`Keywords` : Saint-Venant's Principe, proof, decay, infinite axisymmetrical circular cylinder

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1 Introduction

Saint-Venant's Principle is essential and fundamental in Elasticity (See Ref.[1] and Ref.[2]). Boussinesq and Love announce statements of Saint-Venant's Principe (See Ref.[3] and Ref.[4]), but Mises and Sternberg argue, by citing counterexamples, that the statements are not clear, suggesting that Saint-Venant's Principle should be proved or given a mathematical formulation (See Ref.[5] and Ref.[6]). Truesdell asserts that if Saint-Venant's Principle of equipollent loads is true, it must be a mathematical consequence of the general equations of Linear Elasticity (See Ref.[7]).

There is no doubt that mathematical proof of Saint-Venant's Principle has become an academic attraction for contributors and much effort has been made for exploring its mysterious implications or deciphering its puzzle. Zanaboni "proved" a theorem trying to concern Saint-Venant's Principle in terms of work and energy (See Refs.[8],[9],[10]). However, Zhao argues that Zanaboni's theorem is false (See Ref.[11]). The work published by Toupin cites more counterexamples to explain that Love's statement is false, and then establishes a formulation of energy decay, which is considered as "a precise mathematical formulation and proof" of Saint-Venant's Principle for the elastic cylinder (See Refs.[12], [13]). Furthermore, Toupin's work seems to set up an example followed by a large number of papers to establish Toupin-type energy decay formulae for branches of continuum mechanics. Since 1965 the concept of energy decay suggested by Toupin has been widely accepted by authors, and various techniques have been developed to construct inequalities of Toupin-type decay of energy which are spread widely in continuum mechanics. Especially, the theorem given by Berdichevskii is considered as a generalization of Toupin's theorem (See Ref.[14]). Horgan and Knowles reviewed the development (See Refs.[15],[16],[17]). However, Zhao points out that Toupin's theory is not a strict mathematical proof, and Toupin's Theorem is not an exact mathematical formulation, of Saint-Venant's Principle. Interestingly and significantly, Saint-Venant's Principle stated by Love is disproved mathematically from Toupin's Theorem, so Toupin's Theorem is mathematically inconsistent with Saint-Venant's Principle (See Ref.[11]).

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Boussinesq and Love and points out that Special Saint-Venant's Principle or Modified Saint-Venant's Principle can be proved or formulated (See Ref.[11]).

Saint-Venant's Principle is applied without proof here and there in the literature of Elasticity. It is essential to supplement the literature with mathematical proof or formulation of Special Saint-Venant's Principle or Modified Saint-Venant's Principle of elastic problems one by one unless Elasticity is not to be constructed to be rational, logical, rigorous and secure mechanics.

Here we discuss the problem of Saint-Venant's Principle for the axisymmetrical circular cylinder and establish its formulation of Saint-Venant's decay.

2 Love's Statement of Saint-Venant's Principle and Its Provability

Love's Statement : "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." (See Ref.[4])

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Love, but argues by mathematical analysis that Saint-Venant's decay of strains (then stresses) described by Love's statement can be proved true by special formulating or adding supplementary conditions to the problems discussed (See Ref.[11]).

3 Formulating the Problem

Let us consider a axisymmetrical cylinder of length L with a constant circular cross section (See Ref. [18]). The end ($z = 0$) of the cylinder is loaded by an equilibrium system of forces, otherwise the cylinder would be free. Denoting by a the radius of the circular cross section, the boundary conditions are:

$$r = a \ : \quad \sigma_r = 0, \quad \tau_{rz} = 0, \tag{1}$$

$$z = L \ : \quad \sigma_z = 0, \quad \tau_{zr} = 0. \tag{2}$$

4 Eigenvalue Equation

Let the stress function be

$$\phi(r,z) = \sum_{n=1}^{\infty} \varphi_n(r) e^{-k_n z}, \quad (0 \le r \le a, \ 0 \le z \le L) \tag{3}$$

Putting Eq.(3) into

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}) \ \phi(r,z) = 0, \tag{4}$$

we find the solution

$$\phi(r,z) = \sum_{n=1}^{\infty} \varphi_n(r) e^{-k_n z} = \sum_{n=1}^{\infty} [A_n J_0(k_n r) + B_n k_n r J_1(k_n r)] e^{-k_n z}. \tag{5}$$

From Eq.(5)

$$\begin{aligned} \sigma_r &= \frac{\partial}{\partial z}[\nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2}] \\ &= \sum_{n=1}^{\infty} \{A_n[-k_n^3 J_0(k_n r) + k_n^2 \frac{1}{r} J_1(k_n r)] \\ &+ B_n[(1-2\nu) k_n^3 J_0(k_n r) - k_n^4 r J_1(k_n r)]\} e^{-k_n z}, \end{aligned} \tag{6}$$

$$\begin{aligned}
\tau_{rz} &= \frac{\partial}{\partial r}[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}] \\
&= \sum_{n=1}^{\infty}\{-A_n k_n^3 J_1(k_n r) \\
&- B_n[2(1-\nu)k_n^3 J_1(k_n r) + k_n^4 r J_0(k_n r)]\}e^{-k_n z}. \qquad (7)
\end{aligned}$$

The condition (1) must be fulfilled by (6) and (7), therefore it is required that

$$A_n[-k_n^3 J_0(k_n a) + k_n^2 \frac{1}{a} J_1(k_n a)] + B_n[(1-2\nu)k_n^3 J_0(k_n a) - k_n^4 a J_1(k_n a)] = 0, \quad (8)$$

$$-A_n k_n^3 J_1(k_n a) - B_n[2(1-\nu)k_n^3 J_1(k_n a) + k_n^4 a J_0(k_n a)] = 0. \quad (9)$$

The coefficients A_n and B_n have non-zero solutions from Eqs.(8) and (9) if

$$\begin{aligned}
[& \quad -k_n^3 J_0(k_n a) + k_n^2 \frac{1}{a} J_1(k_n a)][2(1-\nu)k_n^3 J_1(k_n a) + k_n^4 a J_0(k_n a) \quad] \\
- & \quad k_n^3 J_1(k_n a)[(1-2\nu)k_n^3 J_0(k_n a) - k_n^4 a J_1(k_n a)] \\
= & \quad 0. \qquad (10)
\end{aligned}$$

Excluding $k_n = 0$, Eq.(10) is changed into

$$2(1-\nu)k_n^2 a[J_0(k_n a)]^2 + (1-2\nu)k_n J_0(k_n a)J_1(k_n a) - [2(1-\nu)\frac{1}{a} + k_n^2 a][J_1(k_n a)]^2 = 0. \quad (11)$$

Equation (11) is the eigenvalue equation of the problem, from which k_n are determined.

5 Stress Components

From Eq.(5) we find the stress components

$$\begin{aligned}
\sigma_\theta &= \frac{\partial}{\partial z}[\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}] \\
&= \sum_{n=1}^{\infty}[-A_n \frac{1}{r} k_n^2 J_1(k_n r) + B_n(1-2\nu)k_n^3 J_0(k_n r)]e^{-k_n z}, \qquad (12)
\end{aligned}$$

$$
\begin{aligned}
\sigma_z &= \frac{\partial}{\partial z}[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}] \\
&= \sum_{n=1}^{\infty}\{A_n k_n^3 J_0(k_n r) \\
&+ B_n[-2(2-\nu)k_n^3 J_0(k_n r) + k_n^4 r J_1(k_n r)]\}e^{-k_n z}. \qquad (13)
\end{aligned}
$$

6 Saint-Venant's Decay of Stresses and Its Requirement

Condition (2) must be satisfied by Eq.(7) and Eq.(13), and it is required that

$$L \to \infty. \qquad (14)$$

That is to say that the cylinder should be infinitely long. Equation (14) is the requirement for Saint-Venant's decay of stresses, Eqs.(6), (7), (12) and (13), of the axisymmetrical circular cylinder submitted to the equilibrium system of forces defined on the end ($z = 0$) by

$$
\begin{aligned}
\sigma_z &= \sum_{n=1}^{\infty}\{A_n k_n^3 J_0(k_n r) \\
&+ B_n[-2(2-\nu)k_n^3 J_0(k_n r) + k_n^4 r J_1(k_n r)]\}, \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_{zr} &= \sum_{n=1}^{\infty}\{-A_n k_n^3 J_1(k_n r) \\
&- B_n[2(1-\nu)k_n^3 J_1(k_n r) + k_n^4 r J_0(k_n r)]\}. \qquad (16)
\end{aligned}
$$

7 Saint-Venant's Principle of the Axisymmetrical Circular Cylinder

From Eqs.(6), (7), (12) and (13) we have

$$
\begin{aligned}
\lim_{z\to\infty} \sigma_r &= 0, \\
\lim_{z\to\infty} \sigma_\theta &= 0, \\
\lim_{z\to\infty} \sigma_z &= 0, \\
\lim_{z\to\infty} \tau_{rz} &= 0. \qquad (17)
\end{aligned}
$$

We prove Saint-Venant's Principle of the Problem of the Axisymmetrical Circular Cylinder by the end equations in terms of Eq.(17).

8 Conclusion

The Special Saint-Venant's Principle for the problem of the infinite axisymmetrical circular cylinder is proved.

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