Electronic Supplementary Materials

Analytical Solutions of the Displacement and Stress Fields of the Generic Nanocomposite Structure of Biological Materials

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Appendix A: The unknown functions of the initial solutions

The unknown functions $f_i^{(0)}(\xi)$, $g_i^{(0)}(\xi)$, $\lambda_i^{(0)}(\xi)$, $\chi_i^{(0)}(\xi)$, $\kappa^{(0)}(\xi)$, $p^{(0)}(\xi)$, $q^{(0)}(\xi)$, and $s^{(0)}(\xi)$ in the initial solutions (Eq.(22) and Eq.(25)) can be determined by applying appropriate boundary conditions. Firstly, based on the continuity conditions ($v_I^{m(0)} = v^{p(0)}$, $u_I^{m(0)} = u^{p(0)}$, $\sigma_{xy,I}^{m(0)} = \sigma_{xy}^{p(0)}$, $\sigma_{yy,I}^{m(0)} = \sigma_{yy}^{p(0)}$) at the interface between mineral I and the protein layer at $\eta = d_p/(2d_m)$, we can obtain the following relations,

$$\begin{aligned}&-\upsilon_m\left(\frac{d_p}{2d_m}\right)f_I^{(0)\prime}(\xi)+g_I^{(0)}(\xi)=\\&\frac{-l(1+\upsilon_p)^2}{2E_mC_E}\left(\frac{d_p}{2d_m}\right)^2\kappa^{(0)\prime}(\xi)+\frac{l(1-\upsilon_p^2)}{E_mC_E}\left(\frac{d_p}{2d_m}\right)s^{(0)}(\xi)-\upsilon_p\left(\frac{d_p}{2d_m}\right)p^{(0)\prime}(\xi)+q^{(0)}(\xi)\end{aligned} \tag{A1}$$

$$f_I^{(0)}(\xi)=\frac{l}{G_mC_G}\left(\frac{d_p}{2d_m}\right)\kappa^{(0)}(\xi)+p^{(0)}(\xi) \tag{A2}$$

$$-\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right)f_I^{(0)\prime\prime}(\xi)+\lambda_I^{(0)}(\xi)=\kappa^{(0)}(\xi) \tag{A3}$$

$$\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2 f_I^{(0)\prime\prime\prime}(\xi)-\left(\frac{d_p}{2d_m}\right)\lambda_I^{(0)\prime}(\xi)+\chi_I^{(0)}(\xi)=-\left(\frac{d_p}{2d_m}\right)\kappa^{(0)\prime}(\xi)+s^{(0)}(\xi) \tag{A4}$$

Based on the continuity conditions ($v_{II}^{m(0)} = v^{p(0)}, u_{II}^{m(0)} = u^{p(0)}, \sigma_{xy,II}^{m(0)} = \sigma_{xy}^{p(0)}$, $\sigma_{yy,II}^{m(0)} = \sigma_{yy}^{p(0)}$) at the interface between mineral II and the protein layer at $\eta = -d_p/(2d_m)$, we have,

$$\upsilon_m\left(\frac{d_p}{2d_m}\right)f_{II}^{(0)\prime}(\xi)+g_{II}^{(0)}(\xi)=$$

$$\frac{-l\left(1+\upsilon_p\right)^2}{2E_mC_E}\left(\frac{d_p}{2d_m}\right)^2\kappa^{(0)\prime}(\xi)-\frac{l\left(1-\upsilon_p^2\right)}{E_mC_E}\left(\frac{d_p}{2d_m}\right)s^{(0)}(\xi)+\upsilon_p\left(\frac{d_p}{2d_m}\right)p^{(0)\prime}(\xi)+q^{(0)}(\xi) \tag{A5}$$

$$f_{II}^{(0)}(\xi)=-\frac{l}{G_mC_G}\left(\frac{d_p}{2d_m}\right)\kappa^{(0)}(\xi)+p^{(0)}(\xi) \tag{A6}$$

$$\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right)f_{II}^{(0)\prime\prime}(\xi)+\lambda_{II}^{(0)}(\xi)=\kappa^{(0)}(\xi) \tag{A7}$$

$$\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2f_{II}^{(0)\prime\prime\prime}(\xi)+\left(\frac{d_p}{2d_m}\right)\lambda_{II}^{(0)\prime}(\xi)+\chi_{II}^{(0)}(\xi)=\left(\frac{d_p}{2d_m}\right)\kappa^{(0)\prime}(\xi)+s^{(0)}(\xi). \tag{A8}$$

Then, according to the symmetric boundary conditions on AB (see Fig.2), we set $v_I^{m(0)}=C^{(0)}$ and $\sigma_{xy,I}^{m(0)}=0$ at $\eta=\left(d_m+d_p\right)/\left(2d_m\right)$. That is,

$$-\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(0)\prime}(\xi)+g_I^{(0)}(\xi)=C^{(0)} \tag{A9}$$

$$-\frac{E_m}{l}\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(0)\prime\prime}(\xi)+\lambda_I^{(0)}(\xi)=0. \tag{A10}$$

For the symmetric boundary conditions on GH (Fig. 2), we set $v_{II}^{m(0)}=-C^{(0)}$ and $\sigma_{xy,II}^{m(0)}=0$ at $\eta=-\left(d_m+d_p\right)/\left(2d_m\right)$, hence

$$\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)f_{II}^{(0)\prime}(\xi)+g_{II}^{(0)}(\xi)=-C^{(0)} \tag{A11}$$

$$\frac{E_m}{l}\left(\frac{d_m+d_p}{2d_m}\right)f_{II}^{(0)\prime\prime}(\xi)+\lambda_{II}^{(0)}(\xi)=0. \tag{A12}$$

Secondly, we solve the unknown functions with above boundary conditions. Combining Eq. (A7) and Eq. (A3), we obtain

$$-\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right)f_I^{(0)\prime\prime}(\xi)-\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right)f_{II}^{(0)\prime\prime}(\xi)+\lambda_I^{(0)}(\xi)-\lambda_{II}^{(0)}(\xi)=0. \tag{A13}$$

Similarly, subtracting Eq. (A12) from Eq. (A10), we have

$$-\frac{E_m}{l}\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(0)''}(\xi)-\frac{E_m}{l}\left(\frac{d_m+d_p}{2d_m}\right)f_{II}^{(0)''}(\xi)+\lambda_I^{(0)}(\xi)-\lambda_{II}^{(0)}(\xi)=0. \tag{A14}$$

By comparing Eq. (A13) with Eq. (A14), we get

$$f_I^{(0)''}(\xi)=-f_{II}^{(0)''}(\xi). \tag{A15}$$

And substituting Eq. (A15) into Eq. (A10) and Eq. (A12) arrives

$$\lambda_I^{(0)}(\xi)=\lambda_{II}^{(0)}(\xi)=\frac{E_m}{l}\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(0)''}(\xi). \tag{A16}$$

According to Eq. (A16) and Eq. (A3), we obtain

$$\kappa^{(0)}(\xi)=\frac{E_m}{2l}f_I^{(0)''}(\xi). \tag{A17}$$

Next, summation of Eq. (A2) with Eq. (A6) gives

$$f_I^{(0)}(\xi)+f_{II}^{(0)}(\xi)=2p^{(0)}(\xi). \tag{A18}$$

Combining the second order derivatives of Eq. (A18) and Eq.(A15), we have

$$p^{(0)''}(\xi)=0. \tag{A19}$$

which yields,

$$p^{(0)}(\xi)=C_1^{(0)}+C_2^{(0)}\xi. \tag{A20}$$

Substituting Eq. (A17) and Eq. (A20) into Eq. (A2), we have

$$\frac{1}{\gamma^2}f_I^{(0)''}(\xi)-f_I^{(0)}(\xi)=-C_1^{(0)}-C_2^{(0)}\xi. \tag{A21}$$

of which the solution is

$$f_I^{(0)}(\xi)=C_1^{(0)}+C_2^{(0)}\xi+C_3^{(0)}e^{\gamma\xi}+C_4^{(0)}e^{-\gamma\xi} \tag{A22}$$

where $\gamma=l\sqrt{2E_p\Big/\left(E_m d_m d_p\left(1+\upsilon_p\right)\right)}$. Substituting Eq. (A22) into Eq. (A16) and Eq. (A17), respectively, we obtain,

$$\lambda_I^{(0)}\left(\xi\right)=\lambda_{II}^{(0)}\left(\xi\right)=\frac{2G_mC_G\left(d_m+d_p\right)}{ld_p}\left(C_3^{(0)}e^{\gamma\xi}+C_4^{(0)}e^{-\gamma\xi}\right) \tag{A23}$$

and

$$\kappa^{(0)}\left(\xi\right)=\frac{2G_mC_Gd_m}{ld_p}\left(C_3^{(0)}e^{\gamma\xi}+C_4^{(0)}e^{-\gamma\xi}\right). \tag{A24}$$

Substituting Eq. (A20) and Eq. (A24) into Eq. (A6) yields

$$f_{II}^{(0)}\left(\xi\right)=C_1^{(0)}+C_2^{(0)}\xi-C_3^{(0)}e^{\gamma\xi}-C_4^{(0)}e^{-\gamma\xi}. \tag{A25}$$

Substituting Eq. (A22) into Eq. (A9), we have

$$\begin{aligned} g_I^{(0)}\left(\xi\right)&=\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(0)\prime}\left(\xi\right)+C^{(0)}\\ &=\gamma\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)\left(C_3^{(0)}e^{\gamma\xi}-C_4^{(0)}e^{-\gamma\xi}\right)+\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)C_2^{(0)}+C^{(0)} \end{aligned} \tag{A26}$$

and substituting Eq. (A25) into Eq. (A11), we obtain

$$\begin{aligned} g_{II}^{(0)}\left(\xi\right)&=-\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)f_{II}^{(0)\prime}\left(\xi\right)-C^{(0)}\\ &=\gamma\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)\left(C_3^{(0)}e^{\gamma\xi}-C_4^{(0)}e^{-\gamma\xi}\right)-\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)C_2^{(0)}-C^{(0)}. \end{aligned} \tag{A27}$$

Then, subtracting Eq. (A5) from Eq. (A1), we obtain

$$s^{(0)}\left(\xi\right)=\frac{2E_mC_Ed_m}{l\left(1-\upsilon_p^2\right)d_p}C^{(0)}+\frac{E_mC_E\left(\upsilon_md_m+\upsilon_pd_p\right)}{l\left(1-\upsilon_p^2\right)d_p}C_2^{(0)}. \tag{A28}$$

and adding Eq. (A1) to Eq. (A5), we get

$$q^{(0)}\left(\xi\right)=\left(\frac{\upsilon_m}{2}+\frac{\left(1+\upsilon_p\right)d_p}{8d_m}\right)\gamma\left(C_3^{(0)}e^{\gamma\xi}-C_4^{(0)}e^{-\gamma\xi}\right). \tag{A29}$$

Based on Eqs. (A4), (A17), (A22), (A23), (A28), we obtain

$$
\begin{aligned}
\chi_I^{(0)}(\xi) &= -\left(\frac{d_p}{2d_m}\right)\kappa^{(0)\prime}(\xi)+s^{(0)}(\xi)-\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2 f_I^{(0)\prime\prime\prime}(\xi)+\left(\frac{d_p}{2d_m}\right)\lambda_I^{(0)\prime}(\xi) \\
&= \frac{G_m C_G d_p}{2ld_m}\gamma\left(C_3^{(0)}e^{\gamma\xi}-C_4^{(0)}e^{-\gamma\xi}\right)+\frac{2E_m C_E d_m}{ld_p\left(1-\upsilon_p^2\right)}C^{(0)}+\frac{E_m C_E\left(\upsilon_m d_m+\upsilon_p d_p\right)}{ld_p\left(1-\upsilon_p^2\right)}C_2^{(0)}.
\end{aligned}
\tag{A30}
$$

Based on Eqs. (A8), (A17), (A23), (A25), and (A28), we obtain

$$
\begin{aligned}
\chi_{II}^{(0)}(\xi) &= \left(\frac{d_p}{2d_m}\right)\kappa^{(0)\prime}(\xi)+s^{(0)}(\xi)-\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2 f_{II}^{(0)\prime\prime\prime}(\xi)-\left(\frac{d_p}{2d_m}\right)\lambda_{II}^{(0)\prime}(\xi) \\
&= -\frac{G_m C_G d_p}{2ld_m}\gamma\left(C_3^{(0)}e^{\gamma\xi}-C_4^{(0)}e^{-\gamma\xi}\right)+\frac{2E_m C_E d_m}{ld_p\left(1-\upsilon_p^2\right)}C^{(0)}+\frac{E_m C_E\left(\upsilon_m d_m+\upsilon_p d_p\right)}{ld_p\left(1-\upsilon_p^2\right)}C_2^{(0)}.
\end{aligned}
\tag{A31}
$$

Finally, according to the loading conditions (for instance, displacement loading ${u_0}^{(0)}/2=1$ on boundary BD) and considering the equilibrium of the nanostructure and the anti-symmetry of displacement in mineral I respect to mineral II, we have

$$-u_I^{m(0)}(\xi)=u_{II}^{m(0)}(-\xi) \tag{A32}$$

$$u_I^{m(0)}\Big|_{\xi=\frac{1}{4}}=u_0^{(0)}/2 \tag{A33}$$

$$\int_{d_p/2d_m}^{\left(d_m+d_p\right)/2d_m}\sigma_{xx,I}^{m(0)}d\eta=0 \quad \text{where} \quad \xi=-\frac{1}{4} \tag{A34}$$

$$\int_{-1/4}^{1/4}\sigma_{yy,I}^{m(0)}d\xi=0 \quad \text{where} \quad \eta=\frac{d_m+d_p}{2d_m}. \tag{A35}$$

Substituting the expression for $u_i^{m(0)}$ into Eq. (A32) and Eq. (A33), and also substituting the expressions for $\sigma_{xx,i}^{m(0)}$ and $\sigma_{yy,i}^{m(0)}$ into Eq. (A34) and Eq. (A35), respectively, we can obtain

$$C^{(0)}=-\frac{\left(\upsilon_m d_m+\upsilon_p d_p\right)\gamma sh\frac{\gamma}{4}}{d_m\left(\gamma sh\frac{\gamma}{4}+4ch\frac{\gamma}{4}\right)}u_0^{(0)} \tag{A36}$$

$$C_1^{(0)}=0 \tag{A37}$$

$$C_2^{(0)} = \frac{2\gamma sh\frac{\gamma}{4}}{\gamma sh\frac{\gamma}{4}+4ch\frac{\gamma}{4}}u_0^{(0)} \tag{A38}$$

$$C_3^{(0)} = C_4^{(0)} = \frac{1}{\gamma sh\frac{\gamma}{4}+4ch\frac{\gamma}{4}}u_0^{(0)}, \tag{A39}$$

Then we obtain the expressions of all the unknown function by substituting Eqs. (A36) to (A39) into Eqs. (A22), (A25), (A26), (A27), (A23), (A30), (A31), (A20), (A29), (A24) and (A28)

$$f_i^{(0)}(\xi) = A\cdot\left(2\gamma\xi sh\frac{\gamma}{4} \pm 2ch\gamma\xi\right) \tag{A40}$$

$$g_i^{(0)}(\xi) = A\cdot\left(\left(\frac{d_m+d_p}{d_m}\right)\gamma\upsilon_m sh\gamma\xi \pm \left(\frac{\upsilon_m d_p - \upsilon_p d_p}{d_m}\right)\gamma sh\frac{\gamma}{4}\right) \tag{A41}$$

$$\lambda_i^{(0)}(\xi) = A\frac{E_m\left(d_m+d_p\right)}{ld_m}\gamma^2 ch\gamma\xi \tag{A42}$$

$$\chi_i^{(0)}(\xi) = \pm A\frac{E_m d_p^2}{4ld_m^2}\gamma^3 sh\gamma\xi \tag{A43}$$

$$p^{(0)}(\xi) = 2A\gamma\xi sh\frac{\gamma}{4} \tag{A44}$$

$$q^{(0)}(\xi) = A\cdot\left(\frac{\upsilon_m}{2}+\frac{\left(1+\upsilon_p\right)d_p}{8d_m}\right)2\gamma sh\gamma\xi \tag{A45}$$

$$\kappa^{(0)}(\xi) = A\frac{E_m}{l}\gamma^2 ch\gamma\xi \tag{A46}$$

$$s^{(0)}(\xi) = 0. \tag{A47}$$

Appendix B: The first order perturbation correction

In this section, we solve the first order perturbation correction of the perturbation solutions of the governing equations. According to the perturbation expansions obtained in section 3.2, the governing equations of the first order perturbation (the second expansion term) for the mineral crystal are,

$$\begin{cases} \varepsilon_{xx}^{m(1)} = \dfrac{1}{E_m}\left(\sigma_{xx}^{m(1)} - \upsilon_m \sigma_{yy}^{m(0)}\right) = \dfrac{1}{l}\dfrac{\partial u^{m(1)}}{\partial \xi} \\ \varepsilon_{yy}^{m(1)} = \dfrac{1}{E_m}\left(\sigma_{yy}^{m(0)} - \upsilon_m \sigma_{xx}^{m(1)}\right) = \dfrac{1}{l}\dfrac{\partial v^{m(1)}}{\partial \eta} \\ \varepsilon_{xy}^{m(1)} = \dfrac{1}{2G_m}\sigma_{xy}^{m(1)} \end{cases} \tag{B1}$$

$$\sigma_{xy}^{m(0)} = \frac{G_m}{l}\left(\frac{\partial u^{m(1)}}{\partial \eta} + \frac{\partial v^{m(0)}}{\partial \xi}\right) \tag{B2}$$

$$\begin{cases} \dfrac{\partial \sigma_{xy}^{m(1)}}{\partial \eta} + \dfrac{\partial \sigma_{xx}^{m(1)}}{\partial \xi} = 0 \\ \dfrac{\partial \sigma_{xy}^{m(1)}}{\partial \xi} + \dfrac{\partial \sigma_{yy}^{m(1)}}{\partial \eta} = 0. \end{cases} \tag{B3}$$

and those for the protein are,

$$\begin{cases} \varepsilon_{xx}^{p(1)} = \dfrac{1}{E_m C_E}\left(\sigma_{xx}^{p(1)} - \upsilon_p \sigma_{yy}^{p(1)}\right) = \dfrac{1}{l}\dfrac{\partial u^{p(1)}}{\partial \xi} \\ \varepsilon_{yy}^{p(1)} = \dfrac{1}{E_m C_E}\left(\sigma_{yy}^{p(1)} - \upsilon_p \sigma_{xx}^{p(1)}\right) = \dfrac{1}{l}\dfrac{\partial v^{p(1)}}{\partial \eta} \\ \varepsilon_{xy}^{p(1)} = \dfrac{1}{2G_m C_G}\sigma_{xy}^{p(1)} = \dfrac{1}{2l}\left(\dfrac{\partial u^{p(1)}}{\partial \eta} + \dfrac{\partial v^{p(0)}}{\partial \xi}\right) \end{cases} \tag{B4}$$

$$\begin{cases} \dfrac{\partial \sigma_{xy}^{p(1)}}{\partial \eta} + \dfrac{\partial \sigma_{xx}^{p(0)}}{\partial \xi} = 0 \\ \dfrac{\partial \sigma_{xy}^{p(1)}}{\partial \xi} + \dfrac{\partial \sigma_{yy}^{p(1)}}{\partial \eta} = 0. \end{cases} \tag{B5}$$

Solving above governing equations we can obtain the displacement and stress fields of the mineral and protein of the first order perturbation as below

$$\begin{cases} u_i^{m(1)} = f_i^{(1)}(\xi) + L_i(\xi,\eta) \\ v_i^{m(1)} = -\upsilon_m \eta f_i^{(1)\prime}(\xi) + g_i^{(1)}(\xi) + M_i(\xi,\eta) \\ \sigma_{xx,i}^{m(1)} = \dfrac{E_m}{l} f_i^{(1)\prime}(\xi) + N_i(\xi,\eta) \\ \sigma_{xy,i}^{m(1)} = -\dfrac{E_m}{l} \eta f_i^{(1)\prime\prime}(\xi) + \lambda_i^{(1)}(\xi) + O_i(\xi,\eta) \\ \sigma_{yy,i}^{m(1)} = \dfrac{E_m}{2l} \eta^2 f_i^{(1)\prime\prime\prime}(\xi) - \eta \lambda_i^{(1)\prime}(\xi) + \chi_i^{(1)}(\xi) + P_i(\xi,\eta) \end{cases} \tag{B6}$$

for mineral, where $i = I$ and *II*, standing for mineral *I* and mineral *II*, respectively, and

$$\begin{cases} u^{p(1)} = \dfrac{l}{G_m C_G} \eta \kappa^{(1)}(\xi) + p^{(1)}(\xi) + Q(\xi,\eta) \\ v^{p(1)} = \dfrac{-l\left(1+\upsilon_p\right)^2}{2E_m C_E} \eta^2 \kappa^{(1)\prime}(\xi) + \dfrac{l\left(1-\upsilon_p^2\right)}{E_m C_E} \eta s^{(1)}(\xi) - \upsilon_p \eta p^{(1)\prime}(\xi) + q^{(1)}(\xi) + R(\xi,\eta) \\ \sigma_{xx}^{p(1)} = \left(2+\upsilon_p\right) \eta \kappa^{(1)\prime}(\xi) + \dfrac{E_m C_E}{l} p^{(1)\prime}(\xi) + \upsilon_p s^{(1)}(\xi) + S(\xi,\eta) \\ \sigma_{xy}^{p(1)} = \kappa^{(1)}(\xi) + T(\xi,\eta) \\ \sigma_{yy}^{p(1)} = -\eta \kappa^{(1)\prime}(\xi) + s^{(1)}(\xi) + U(\xi,\eta) \end{cases} \tag{B7}$$

for protein. The unknown functions $L_i(\xi,\eta)$, $M_i(\xi,\eta)$, $N_i(\xi,\eta)$, $O_i(\xi,\eta)$, $P_i(\xi,\eta)$, $Q(\xi,\eta)$, $R(\xi,\eta)$, $S(\xi,\eta)$, $T(\xi,\eta)$, $U(\xi,\eta)$; and $f_i^{(1)}(\xi)$, $g_i^{(1)}(\xi)$, $\lambda_i^{(1)}(\xi)$, $\chi_i^{(1)}(\xi)$, $\kappa^{(1)}(\xi)$, $p^{(1)}(\xi)$, $s^{(1)}(\xi)$, $q^{(1)}(\xi)$ can be determined by applying the boundary conditions.

The unknown functions in Eq.(B6) are

$$L_i(\xi,\eta) = -\frac{2+\upsilon_m}{2} \eta^2 f_i^{(0)\prime\prime}(\xi) - \eta g_i^{(0)\prime}(\xi) + \frac{l}{G_m} \eta \lambda_i^{(0)}(\xi) \tag{B8}$$

$$M_i(\xi,\eta) = \frac{1+2\upsilon_m}{6} \eta^3 f_i^{(0)\prime\prime\prime}(\xi) + \frac{\upsilon_m}{2} \eta^2 g_i^{(0)\prime\prime}(\xi) - \frac{l\left(1+\upsilon_m\right)^2}{2E_m} \eta^2 \lambda_i^{(0)\prime}(\xi) + \frac{l\left(1-\upsilon_m^2\right)}{E_m} \eta \chi_i^{(0)}(\xi) \tag{B9}$$

$$N_i(\xi,\eta) = -\frac{E_m}{l} \eta^2 f_i^{(0)\prime\prime\prime}(\xi) - \frac{E_m}{l} \eta g_i^{(0)\prime\prime}(\xi) + \left(2+\upsilon_m\right) \eta \lambda_i^{(0)\prime}(\xi) + \upsilon_m \chi_i^{(0)}(\xi) \tag{B10}$$

$$O_i(\xi,\eta)=\frac{E_m}{3l}\eta^3 f_i^{(0)''''}(\xi)+\frac{E_m}{2l}\eta^2 g_i^{(0)'''}(\xi)-\frac{2+\upsilon_m}{2}\eta^2\lambda_i^{(0)''}(\xi)-\upsilon_m\eta\chi_i^{(0)'}(\xi) \tag{B11}$$

$$P_i(\xi,\eta)=-\frac{E_m}{12l}\eta^4 f_i^{(0)'''''}(\xi)-\frac{E_m}{6l}\eta^3 g_i^{(0)''''}(\xi)+\frac{2+\upsilon_m}{6}\eta^3\lambda_i^{(0)'''}(\xi)+\frac{\upsilon_m}{2}\eta^2\chi_i^{(0)''}(\xi) \tag{B12}$$

where $i=I$ and II denoting the mineral I and mineral II, respectively.

And the unknown functions in Eq.(B7) are

$$Q(\xi,\eta)=-\frac{l(1+\upsilon_p)(3+\upsilon_p)}{6E_mC_E}\eta^3\kappa^{(0)''}(\xi)-\frac{(2+\upsilon_p)}{2}\eta^2 p^{(0)''}(\xi)-\frac{l(1+\upsilon_p)^2}{2E_mC_E}\eta^2 s^{(0)'}(\xi)-\eta q^{(0)'}(\xi) \tag{B13}$$

$$R(\xi,\eta)=\frac{l(1+\upsilon_p)^2}{12E_mC_E}\eta^4\kappa^{(0)'''}(\xi)+\frac{(1+2\upsilon_p)}{6}\eta^3 p^{(0)'''}(\xi)+\frac{l\upsilon_p(1+\upsilon_p)}{3E_mC_E}\eta^3 s^{(0)''}(\xi)+\frac{\upsilon_p}{2}\eta^2 q^{(0)''}(\xi) \tag{B14}$$

$$S(\xi,\eta)=-\frac{(3+2\upsilon_p)}{6}\eta^3\kappa^{(0)'''}(\xi)-\frac{E_mC_E}{l}\eta^2 p^{(0)'''}(\xi)-\frac{(1+2\upsilon_p)}{2}\eta^2 s^{(0)''}(\xi)-\frac{E_mC_E}{l}\eta q^{(0)''}(\xi) \tag{B15}$$

$$T(\xi,\eta)=-\frac{2+\upsilon_p}{2}\eta^2\kappa^{(0)''}(\xi)-\frac{E_mC_E}{l}\eta p^{(0)''}(\xi)-\upsilon_p\eta s^{(0)'}(\xi) \tag{B16}$$

$$U(\xi,\eta)=\frac{2+\upsilon_p}{6}\eta^3\kappa^{(0)'''}(\xi)+\frac{E_mC_E}{2l}\eta^2 p^{(0)'''}(\xi)+\frac{\upsilon_p}{2}\eta^2 s^{(0)''}(\xi). \tag{B17}$$

Substituting Eqs. (A40) to (A47) into above equations, we have

$$\begin{aligned}
L_i(\xi,\eta)&=-\frac{2+\upsilon_m}{2}\eta^2 f_i^{(0)''}(\xi)-\eta g_i^{(0)'}(\xi)+\frac{l}{G_m}\eta\lambda_i^{(0)}(\xi)\\
&=-\frac{2+\upsilon_m}{2}\eta^2 A\left(\pm 2\gamma^2 ch\gamma\xi\right)-\eta A\left(\frac{d_m+d_p}{d_m}\right)\gamma^2\upsilon_m ch\gamma\xi+\frac{l}{G_m}\eta A\frac{E_m(d_m+d_p)}{ld_m}\gamma^2 ch\gamma\xi\\
&=\frac{2+\upsilon_m}{2}\eta^2 A\left(\mp 2\gamma^2 ch\gamma\xi\right)+\eta A\frac{(2+\upsilon_m)(d_m+d_p)}{d_m}\gamma^2 ch\gamma\xi\\
&=A(2+\upsilon_m)\left(\mp\eta^2+\left(\frac{d_m+d_p}{d_m}\right)\eta\right)\gamma^2 ch\gamma\xi
\end{aligned} \tag{B18}$$

$$
\begin{aligned}
M_i(\xi,\eta) &= \frac{1+2\upsilon_m}{6}\eta^3 f_i^{(0)''''}(\xi) + \frac{\upsilon_m}{2}\eta^2 g_i^{(0)'''}(\xi) - \frac{l(1+\upsilon_m)^2}{2E_m}\eta^2\lambda_i^{(0)'}(\xi) + \frac{l(1-\upsilon_m^2)}{E_m}\eta\chi_i^{(0)}(\xi) \\
&= \frac{1+2\upsilon_m}{6}\eta^3 A\left(\pm 2\gamma^3 sh\gamma\xi\right) + \frac{\upsilon_m}{2}\eta^2 A\left(\frac{d_m+d_p}{d_m}\right)\gamma^3\upsilon_m sh\gamma\xi \\
&\quad - \frac{l(1+\upsilon_m)^2}{2E_m}\eta^2 A\frac{E_m(d_m+d_p)}{ld_m}\gamma^3 sh\gamma\xi + \frac{l(1-\upsilon_m^2)}{E_m}\eta\left(\pm A\frac{E_m d_p^2}{4ld_m^2}\gamma^3 sh\gamma\xi\right) \\
&= A\left(\pm\frac{1+2\upsilon_m}{3}\eta^3 - \frac{d_m+d_p}{d_m}\frac{1+2\upsilon_m}{2}\eta^2 \pm \frac{(1-\upsilon_m^2)d_p^2}{4d_m^2}\eta\right)\gamma^3 sh\gamma\xi
\end{aligned}
\tag{B19}
$$

$$
\begin{aligned}
N_i(\xi,\eta) &= -\frac{E_m}{l}\eta^2 f_i^{(0)''''}(\xi) - \frac{E_m}{l}\eta g_i^{(0)'''}(\xi) + (2+\upsilon_m)\eta\lambda_i^{(0)'}(\xi) + \upsilon_m\chi_i^{(0)}(\xi) \\
&= -\frac{E_m}{l}\eta^2 A\left(\pm 2\gamma^3 sh\gamma\xi\right) - \frac{E_m}{l}\eta A\left(\frac{d_m+d_p}{d_m}\right)\gamma^3\upsilon_m sh\gamma\xi \\
&\quad + (2+\upsilon_m)\eta A\frac{E_m(d_m+d_p)}{ld_m}\gamma^3 sh\gamma\xi + \upsilon_m\left(\pm A\frac{E_m d_p^2}{4ld_m^2}\gamma^3 sh\gamma\xi\right) \\
&= A\left(\mp\frac{2E_m}{l}\eta^2 + \frac{2E_m}{l}\frac{(d_m+d_p)}{d_m}\eta \pm \frac{E_m\upsilon_m d_p^2}{4ld_m^2}\right)\gamma^3 sh\gamma\xi
\end{aligned}
\tag{B20}
$$

$$
\begin{aligned}
O_i(\xi,\eta) &= \frac{E_m}{3l}\eta^3 f_i^{(0)'''''}(\xi) + \frac{E_m}{2l}\eta^2 g_i^{(0)''''}(\xi) - \frac{2+\upsilon_m}{2}\eta^2\lambda_i^{(0)''}(\xi) - \upsilon_m\eta\chi_i^{(0)'}(\xi) \\
&= \frac{E_m}{3l}\eta^3 A\left(\pm 2\gamma^4 ch\gamma\xi\right) + \frac{E_m}{2l}\eta^2 A\left(\frac{d_m+d_p}{d_m}\right)\gamma^4\upsilon_m ch\gamma\xi \\
&\quad - \frac{2+\upsilon_m}{2}\eta^2 A\frac{E_m(d_m+d_p)}{ld_m}\gamma^4 ch\gamma\xi - \upsilon_m\eta\left(\pm A\frac{E_m d_p^2}{4ld_m^2}\gamma^4 ch\gamma\xi\right) \\
&= A\left(\pm\frac{2E_m}{3l}\eta^3 - \frac{E_m}{l}\frac{(d_m+d_p)}{d_m}\eta^2 \mp \frac{E_m\upsilon_m d_p^2}{4ld_m^2}\eta\right)\gamma^4 ch\gamma\xi
\end{aligned}
\tag{B21}
$$

$$
\begin{aligned}
P_i(\xi,\eta) &= -\frac{E_m}{12l}\eta^4 f_i^{(0)''''}(\xi) - \frac{E_m}{6l}\eta^3 g_i^{(0)''''}(\xi) + \frac{2+\upsilon_m}{6}\eta^3\lambda_i^{(0)'''}(\xi) + \frac{\upsilon_m}{2}\eta^2\chi_i^{(0)''}(\xi) \\
&= -\frac{E_m}{12l}\eta^4 A\left(\pm 2\gamma^5 sh\gamma\xi\right) - \frac{E_m}{6l}\eta^3 A\left(\frac{d_m+d_p}{d_m}\right)\gamma^5\upsilon_m sh\gamma\xi \\
&\quad + \frac{2+\upsilon_m}{6}\eta^3 A\frac{E_m\left(d_m+d_p\right)}{ld_m}\gamma^5 sh\gamma\xi + \frac{\upsilon_m}{2}\eta^2\left(\pm A\frac{E_m d_p^2}{4ld_m^2}\gamma^5 sh\gamma\xi\right) \\
&= A\left(\mp\frac{E_m}{6l}\eta^4 + \frac{E_m}{3l}\frac{\left(d_m+d_p\right)}{d_m}\eta^3 \pm \frac{E_m\upsilon_m d_p^2}{8ld_m^2}\eta^2\right)\gamma^5 sh\gamma\xi
\end{aligned}
\tag{B22}
$$

$$
\begin{aligned}
Q(\xi,\eta) &= -\frac{l\left(1+\upsilon_p\right)\left(3+\upsilon_p\right)}{6E_mC_E}\eta^3\kappa^{(0)''}(\xi) - \frac{\left(2+\upsilon_p\right)}{2}\eta^2 p^{(0)''}(\xi) - \frac{l\left(1+\upsilon_p\right)^2}{2E_mC_E}\eta^2 s^{(0)'}(\xi) - \eta q^{(0)'}(\xi) \\
&= -\frac{l\left(1+\upsilon_p\right)\left(3+\upsilon_p\right)}{6E_mC_E}\eta^3 A\frac{E_m}{l}\gamma^4 ch\gamma\xi - \eta A\left(\upsilon_m + \frac{\left(1+\upsilon_p\right)d_p}{4d_m}\right)\gamma^2 ch\gamma\xi \\
&= A\left(-\frac{\left(1+\upsilon_p\right)\left(3+\upsilon_p\right)}{6C_E}\eta^3\gamma^2 - \left(\upsilon_m + \frac{\left(1+\upsilon_p\right)d_p}{4d_m}\right)\eta\right)\gamma^2 ch\gamma\xi
\end{aligned}
\tag{B23}
$$

$$
\begin{aligned}
R(\xi,\eta) &= \frac{l\left(1+\upsilon_p\right)^2}{12E_mC_E}\eta^4\kappa^{(0)'''}(\xi) + \frac{\left(1+2\upsilon_p\right)}{6}\eta^3 p^{(0)'''}(\xi) + \frac{l\upsilon_p\left(1+\upsilon_p\right)}{3E_mC_E}\eta^3 s^{(0)''}(\xi) + \frac{\upsilon_p}{2}\eta^2 q^{(0)''}(\xi) \\
&= \frac{l\left(1+\upsilon_p\right)^2}{12E_mC_E}\eta^4 A\frac{E_m}{l}\gamma^5 sh\gamma\xi + \frac{\upsilon_p}{2}\eta^2 A\left(\upsilon_m + \frac{\left(1+\upsilon_p\right)d_p}{4d_m}\right)\gamma^3 sh\gamma\xi \\
&= A\left(\frac{\left(1+\upsilon_p\right)^2}{12C_E}\eta^4\gamma^2 + \frac{\upsilon_p}{2}\left(\upsilon_m + \frac{\left(1+\upsilon_p\right)d_p}{4d_m}\right)\eta^2\right)\gamma^3 sh\gamma\xi
\end{aligned}
\tag{B24}
$$

$$
\begin{aligned}
S(\xi,\eta) &= -\frac{\left(3+2\upsilon_p\right)}{6}\eta^3\kappa^{(0)'''}(\xi) - \frac{E_mC_E}{l}\eta^2 p^{(0)'''}(\xi) - \frac{\left(1+2\upsilon_p\right)}{2}\eta^2 s^{(0)''}(\xi) - \frac{E_mC_E}{l}\eta q^{(0)''}(\xi) \\
&= -\frac{\left(3+2\upsilon_p\right)}{6}\eta^3 A\frac{E_m}{l}\gamma^5 sh\gamma\xi - \frac{E_mC_E}{l}\eta A\left(\upsilon_m + \frac{\left(1+\upsilon_p\right)d_p}{4d_m}\right)\gamma^3 sh\gamma\xi \\
&= A\left(-\frac{\left(3+2\upsilon_p\right)E_m}{6l}\eta^3\gamma^2 - \frac{E_mC_E}{l}\left(\upsilon_m + \frac{\left(1+\upsilon_p\right)d_p}{4d_m}\right)\eta\right)\gamma^3 sh\gamma\xi
\end{aligned}
\tag{B25}
$$

$$T(\xi,\eta) = -\frac{2+\upsilon_p}{2}\eta^2\kappa^{(0)''}(\xi) - \frac{E_m C_E}{l}\eta p^{(0)''}(\xi) - \upsilon_p \eta s^{(0)'}(\xi)$$
$$= -\frac{2+\upsilon_p}{2}\eta^2 A\frac{E_m}{l}\gamma^4 ch\gamma\xi \tag{B26}$$
$$= -A\frac{(2+\upsilon_p)E_m}{2l}\eta^2\gamma^4 ch\gamma\xi$$

$$U(\xi,\eta) = \frac{2+\upsilon_p}{6}\eta^3\kappa^{(0)'''}(\xi) + \frac{E_m C_E}{2l}\eta^2 p^{(0)'''}(\xi) + \frac{\upsilon_p}{2}\eta^2 s^{(0)''}(\xi)$$
$$= \frac{2+\upsilon_p}{6}\eta^3 A\frac{E_m}{l}\gamma^5 sh\gamma\xi \tag{B27}$$
$$= A\frac{(2+\upsilon_p)E_m}{6l}\eta^3\gamma^5 sh\gamma\xi.$$

It can be found that

$$L_I(\xi,-\eta) = -L_{II}(\xi,\eta) \tag{B28}$$

$$M_I(\xi,-\eta) = M_{II}(\xi,\eta) \tag{B29}$$

$$N_I(\xi,-\eta) = -N_{II}(\xi,\eta) \tag{B30}$$

$$O_I(\xi,-\eta) = O_{II}(\xi,\eta) \tag{B31}$$

$$P_I(\xi,-\eta) = -P_{II}(\xi,\eta) \tag{B32}$$

$$Q(\xi,-\eta) = -Q(\xi,\eta) \tag{B33}$$

$$R(\xi,-\eta) = R(\xi,\eta) \tag{B34}$$

$$S(\xi,-\eta) = -S(\xi,\eta) \tag{B35}$$

$$T(\xi,-\eta) = T(\xi,\eta) \tag{B36}$$

$$U(\xi,-\eta) = -U(\xi,\eta). \tag{B37}$$

In the following, we will determine the coefficients in above unknown functions. Firstly, according to the continuity conditions ($u_I^{m(1)} = u^{p(1)}, v_I^{m(1)} = v^{p(1)}$, $\sigma_{xy,I}^{m(1)} = \sigma_{xy}^{p(1)}$, $\sigma_{yy,I}^{m(1)} = \sigma_{yy}^{p(1)}$) on the interface of the mineral *I* and the protein at $\eta = d_p/(2d_m)$, we have

$$-\upsilon_m\left(\frac{d_p}{2d_m}\right)f_I^{(1)\prime}(\xi)+g_I^{(1)}(\xi)+M_I\left(\xi,\frac{d_p}{2d_m}\right)-R\left(\xi,\frac{d_p}{2d_m}\right)=$$
$$\frac{-l\left(1+\upsilon_p\right)^2}{2E_mC_E}\left(\frac{d_p}{2d_m}\right)^2\kappa^{(1)\prime}(\xi)+\frac{l\left(1-\upsilon_p^2\right)}{E_mC_E}\left(\frac{d_p}{2d_m}\right)s^{(1)}(\xi)-\upsilon_p\left(\frac{d_p}{2d_m}\right)p^{(1)\prime}(\xi)+q^{(1)}(\xi) \tag{B38}$$

$$f_I^{(1)}(\xi)+L_I\left(\xi,\frac{d_p}{2d_m}\right)=\frac{l}{G_mC_G}\left(\frac{d_p}{2d_m}\right)\kappa^{(1)}(\xi)+p^{(1)}(\xi)+Q\left(\xi,\frac{d_p}{2d_m}\right) \tag{B39}$$

$$-\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right)f_I^{(1)\prime\prime}(\xi)+\lambda_I^{(1)}(\xi)+O_I\left(\xi,\frac{d_p}{2d_m}\right)=\kappa^{(1)}(\xi)+T\left(\xi,\frac{d_p}{2d_m}\right) \tag{B40}$$

$$\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2f_I^{(1)\prime\prime\prime}(\xi)-\left(\frac{d_p}{2d_m}\right)\lambda_I^{(1)\prime}(\xi)+\chi_I^{(1)}(\xi)+P_I\left(\xi,\frac{d_p}{2d_m}\right)$$
$$=-\left(\frac{d_p}{2d_m}\right)\kappa^{(1)\prime}(\xi)+s^{(1)}(\xi)+U\left(\xi,\frac{d_p}{2d_m}\right). \tag{B41}$$

And according to the continuity condition ($u_{II}^{m(1)}=u^{p(1)}, v_{II}^{m(1)}=v^{p(1)}$, $\sigma_{xy,II}^{m(1)}=\sigma_{xy}^{p(1)}$, $\sigma_{yy,II}^{m(1)}=\sigma_{yy}^{p(1)}$) on the interface of the mineral *II* and the protein at $\eta=-d_p/(2d_m)$, we obtain

$$\upsilon_m\left(\frac{d_p}{2d_m}\right)f_{II}^{(1)\prime}(\xi)+g_{II}^{(1)}(\xi)+M_{II}\left(\xi,-\frac{d_p}{2d_m}\right)-R\left(\xi,-\frac{d_p}{2d_m}\right)=$$
$$\frac{-l\left(1+\upsilon_p\right)^2}{2E_mC_E}\left(\frac{d_p}{2d_m}\right)^2\kappa^{(1)\prime}(\xi)-\frac{l\left(1-\upsilon_p^2\right)}{E_mC_E}\left(\frac{d_p}{2d_m}\right)s^{(1)}(\xi)+\upsilon_p\left(\frac{d_p}{2d_m}\right)p^{(1)\prime}(\xi)+q^{(1)}(\xi) \tag{B42}$$

$$f_{II}^{(1)}(\xi)+L_{II}\left(\xi,-\frac{d_p}{2d_m}\right)=\frac{-l}{G_mC_G}\left(\frac{d_p}{2d_m}\right)\kappa^{(1)}(\xi)+p^{(1)}(\xi)+Q\left(\xi,-\frac{d_p}{2d_m}\right) \tag{B43}$$

$$\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right)f_{II}^{(1)\prime\prime}(\xi)+\lambda_{II}^{(1)}(\xi)+O_{II}\left(\xi,-\frac{d_p}{2d_m}\right)=\kappa^{(1)}(\xi)+T\left(\xi,-\frac{d_p}{2d_m}\right) \tag{B44}$$

$$\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2f_{II}^{(1)\prime\prime\prime}(\xi)+\left(\frac{d_p}{2d_m}\right)\lambda_{II}^{(1)\prime}(\xi)+\chi_{II}^{(1)}(\xi)+P_{II}\left(\xi,-\frac{d_p}{2d_m}\right)$$
$$=\left(\frac{d_p}{2d_m}\right)\kappa^{(1)\prime}(\xi)+s^{(1)}(\xi)+U\left(\xi,-\frac{d_p}{2d_m}\right). \tag{B45}$$

And from the symmetric boundary conditions on AB (see Fig. 2), we set $v_I^{m(1)}=C^{(1)}$ and $\sigma_{xy,I}^{m(1)}=0$ at

$\eta = \left(d_m + d_p\right) \big/ \left(2d_m\right)$, then we have

$$-\upsilon_m \left(\frac{d_m + d_p}{2d_m}\right) f_I^{(1)\prime}\left(\xi\right) + g_I^{(1)}\left(\xi\right) + M_I \left(\xi, \frac{d_m + d_p}{2d_m}\right) = C^{(1)} \tag{B46}$$

$$-\frac{E_m}{l}\left(\frac{d_m + d_p}{2d_m}\right) f_I^{(1)\prime\prime}\left(\xi\right) + \lambda_I^{(1)}\left(\xi\right) + O_I \left(\xi, \frac{d_m + d_p}{2d_m}\right) = 0. \tag{B47}$$

And for the symmetric boundary conditions on GH (Fig. 2), we set $v_{II}^{m(1)} = -C^{(1)}$ and $\sigma_{xy,II}^{m(1)} = 0$, at $\eta = -\left(d_m + d_p\right) \big/ \left(2d_m\right)$, then we have

$$\upsilon_m \left(\frac{d_m + d_p}{2d_m}\right) f_{II}^{(1)\prime}\left(\xi\right) + g_{II}^{(1)}\left(\xi\right) + M_{II} \left(\xi, -\frac{d_m + d_p}{2d_m}\right) = -C^{(1)} \tag{B48}$$

$$\frac{E_m}{l}\left(\frac{d_m + d_p}{2d_m}\right) f_{II}^{(1)\prime\prime}\left(\xi\right) + \lambda_{II}^{(1)}\left(\xi\right) + O_{II} \left(\xi, -\frac{d_m + d_p}{2d_m}\right) = 0. \tag{B49}$$

Secondly, we use above relations obtained from the boundary conditions and continuity conditions to solve the unknown functions. Subtracting Eq. (B40) from Eq. (B44), and applying $O_I\left(\xi, -\eta\right) = O_{II}\left(\xi, \eta\right)$, $T\left(\xi, -\eta\right) = T\left(\xi, \eta\right)$, we obtain

$$-\frac{E_m}{l}\left(\frac{d_p}{2d_m}\right) f_I^{(1)\prime\prime}\left(\xi\right) - \frac{E_m}{l}\left(\frac{d_p}{2d_m}\right) f_{II}^{(1)\prime\prime}\left(\xi\right) + \lambda_I^{(1)}\left(\xi\right) - \lambda_{II}^{(1)}\left(\xi\right) = 0. \tag{B50}$$

And subtracting Eq. (B47) from Eq. (B49) and applying $O_I\left(\xi, -\eta\right) = O_{II}\left(\xi, \eta\right)$, we obtain

$$-\frac{E_m}{l}\left(\frac{d_m + d_p}{2d_m}\right) f_I^{(1)\prime\prime}\left(\xi\right) - \frac{E_m}{l}\left(\frac{d_m + d_p}{2d_m}\right) f_{II}^{(1)\prime\prime}\left(\xi\right) + \lambda_I^{(1)}\left(\xi\right) - \lambda_{II}^{(1)}\left(\xi\right) = 0. \tag{B51}$$

Combining Eq. (B50) and (B51) leads to

$$f_I^{(1)\prime\prime}\left(\xi\right) + f_{II}^{(1)\prime\prime}\left(\xi\right) = 0. \tag{B52}$$

Then substituting Eq. (B52) into Eq. (B47) and Eq. (B49), and considering $O_I\left(\xi, -\eta\right) = O_{II}\left(\xi, \eta\right)$, we obtain

$$\lambda_I^{(1)}\left(\xi\right) = \lambda_{II}^{(1)}\left(\xi\right) = \frac{E_m}{l}\left(\frac{d_m + d_p}{2d_m}\right) f_I^{(1)\prime\prime}\left(\xi\right) - O_I \left(\xi, \frac{d_m + d_p}{2d_m}\right). \tag{B53}$$

And substituting Eq. (B53) into Eq. (B40), we obtain

$$\kappa^{(1)}(\xi)=\frac{E_m}{2l}f_I^{(1)''}(\xi)+O_I\left(\xi,\frac{d_p}{2d_m}\right)-O_I\left(\xi,\frac{d_m+d_p}{2d_m}\right)-T\left(\xi,\frac{d_p}{2d_m}\right). \tag{B54}$$

Next, by adding Eq. (B39) to Eq. (B43), and considering the relationship $L_I(\xi,-\eta)=-L_{II}(\xi,\eta)$, we obtain

$$f_I^{(1)}(\xi)+f_{II}^{(1)}(\xi)=2p^{(1)}(\xi). \tag{B55}$$

According to Eq. (B52) and (B55), we have

$$2p^{(1)''}(\xi)=0. \tag{B56}$$

of which the general solution is

$$p^{(1)}(\xi)=C_1^{(1)}+C_2^{(1)}\xi. \tag{B57}$$

Substituting Eq. (B54) and Eq. (B57) into Eq. (B39), we obtain

$$\begin{aligned}&\frac{1}{\gamma^2}f_I^{(1)''}(\xi)-f_I^{(1)}(\xi)=-C_1^{(1)}-C_2^{(1)}\xi+L_I\left(\xi,\frac{d_p}{2d_m}\right)-Q\left(\xi,\frac{d_p}{2d_m}\right)\\&-\frac{2l}{E_m\gamma^2}\left(O_I\left(\xi,\frac{d_p}{2d_m}\right)-O_I\left(\xi,\frac{d_m+d_p}{2d_m}\right)-T\left(\xi,\frac{d_p}{2d_m}\right)\right).\end{aligned} \tag{B58}$$

of which the general solution is

$$\begin{aligned}f_I^{(1)}(\xi)=&C_1^{(1)}+C_2^{(1)}\xi+C_3^{(1)}e^{\gamma\xi}+C_4^{(1)}e^{-\gamma\xi}\\&-\frac{AB\gamma^2}{48d_m^3}\left(e^{-\gamma\xi}\left(ch\gamma\xi sh\gamma\xi+ch^2\gamma\xi+\gamma\xi\right)-e^{\gamma\xi}\left(ch\gamma\xi sh\gamma\xi-\frac{ch2\gamma\xi}{2}+\gamma\xi\right)\right)\end{aligned} \tag{B59}$$

where $A=u_0^{(0)}\Big/\left(\gamma sh\frac{\gamma}{4}+4ch\frac{\gamma}{4}\right)$, $B=12\upsilon_m d_p d_m^2-3d_m d_p^2-\upsilon_p d_m d_p^2-4d_m^3$.

From Eq. (B55) and Eq. (B57), we obtain

$$\begin{aligned}f_{II}^{(1)}(\xi)=&C_1^{(1)}+C_2^{(1)}\xi-C_3^{(1)}e^{\gamma\xi}-C_4^{(1)}e^{-\gamma\xi}\\&+\frac{AB\gamma^2}{48d_m^3}\left(e^{-\gamma\xi}\left(ch\gamma\xi sh\gamma\xi+ch^2\gamma\xi+\gamma\xi\right)-e^{\gamma\xi}\left(ch\gamma\xi sh\gamma\xi-\frac{ch2\gamma\xi}{2}+\gamma\xi\right)\right).\end{aligned} \tag{B60}$$

From Eqs. (B46), (B48), (B53), (B54), (B57), (B41) and (B45), we obtain

$$g_I^{(1)}(\xi)=\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(1)\prime}(\xi)-M_I\left(\xi,\frac{d_m+d_p}{2d_m}\right)+C^{(1)} \tag{B61}$$

$$g_{II}^{(1)}(\xi)=-\upsilon_m\left(\frac{d_m+d_p}{2d_m}\right)f_{II}^{(1)\prime}(\xi)-M_{II}\left(\xi,-\frac{d_m+d_p}{2d_m}\right)-C^{(1)} \tag{B62}$$

$$\lambda_I^{(1)}(\xi)=\lambda_{II}^{(1)}(\xi)=\frac{E_m}{l}\left(\frac{d_m+d_p}{2d_m}\right)f_I^{(1)\prime\prime}(\xi)-O_I\left(\xi,\frac{d_m+d_p}{2d_m}\right) \tag{B63}$$

$$\kappa^{(1)}(\xi)=\frac{E_m}{2l}f_I^{(1)\prime\prime}(\xi)+O_I\left(\xi,\frac{d_p}{2d_m}\right)-O_I\left(\xi,\frac{d_m+d_p}{2d_m}\right)-T\left(\xi,\frac{d_p}{2d_m}\right) \tag{B64}$$

$$p^{(1)}(\xi)=C_1^{(1)}+C_2^{(1)}\xi \tag{B65}$$

$$\begin{aligned}\chi_I^{(1)}(\xi)=&-\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2 f_I^{(1)\prime\prime\prime}(\xi)-\left(\frac{d_p}{2d_m}\right)\kappa^{(1)\prime}(\xi)+\left(\frac{d_p}{2d_m}\right)\lambda_I^{(1)\prime}(\xi)\\&+s^{(1)}(\xi)+U\left(\xi,\frac{d_p}{2d_m}\right)-P_I\left(\xi,\frac{d_p}{2d_m}\right)\end{aligned} \tag{B66}$$

$$\begin{aligned}\chi_{II}^{(1)}(\xi)=&-\frac{E_m}{2l}\left(\frac{d_p}{2d_m}\right)^2 f_{II}^{(1)\prime\prime\prime}(\xi)+\left(\frac{d_p}{2d_m}\right)\kappa^{(1)\prime}(\xi)-\left(\frac{d_p}{2d_m}\right)\lambda_{II}^{(1)\prime}(\xi)\\&+s^{(1)}(\xi)+U\left(\xi,-\frac{d_p}{2d_m}\right)-P_{II}\left(\xi,-\frac{d_p}{2d_m}\right).\end{aligned} \tag{B67}$$

Then subtracting Eq. (B38) from Eq. (B42), and considering the relationship $M_I(\xi,-\eta)=M_{II}(\xi,\eta)$ and $R(\xi,-\eta)=R(\xi,\eta)$, we obtain

$$s^{(1)}(\xi)=\frac{E_m C_E d_m}{l\left(1-\upsilon_p^2\right)d_p}\left(\frac{\left(\upsilon_m d_m+\upsilon_p d_p\right)}{d_m}C_2^{(1)}+2C^{(1)}\right). \tag{B68}$$

By adding Eq. (B38) to Eq. (B42), and considering the relationship $M_I(\xi,-\eta)=M_{II}(\xi,\eta)$ and $R(\xi,-\eta)=R(\xi,\eta)$, we obtain

$$q^{(1)}(\xi)=\frac{\upsilon_m}{4}\left(f_I^{(1)\prime}(\xi)-f_{II}^{(1)\prime}(\xi)\right)+M_I\left(\xi,\frac{d_p}{2d_m}\right)-M_I\left(\xi,\frac{d_m+d_p}{2d_m}\right)$$
$$-R\left(\xi,\frac{d_p}{2d_m}\right)+\frac{l\left(1+\upsilon_p\right)^2}{2E_mC_E}\left(\frac{d_p}{2d_m}\right)^2\kappa^{(1)\prime}(\xi). \tag{B69}$$

Finally, the unknown coefficients $C^{(1)}$, $C_1^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}$ and $C_4^{(1)}$ in above equations can be obtained according to the loading conditions (For instance, displacement loading $u_0^{(1)}/2=0$ on boundary BD) and considering the equilibrium of the nanostructure and the anti-symmetry of displacement in mineral I respect to mineral II, which are

$$-u_I^{m(1)}(\xi,\eta)=u_{II}^{m(1)}(-\xi,-\eta) \tag{B70}$$

$$u_I^{m(1)}\left(\frac{1}{4},\frac{d_p+d_m}{2d_m}\right)=u_0^{(1)}/2 \tag{B71}$$

$$\int_{d_p/2d_m}^{(d_m+d_p)/2d_m}\sigma_{xx,I}^{m(1)}d\eta=0 \quad \text{where} \quad \xi=-1/4 \tag{B72}$$

$$\int_{-1/4}^{1/4}\sigma_{yy,I}^{m(1)}d\xi=0 \quad \text{where} \quad \eta=\left(d_m+d_p\right)/\left(2d_m\right). \tag{B73}$$

which can be further expressed as below,

$$-f_I^{(1)}(\xi)-L_I(\xi,\eta)=f_{II}^{(1)}(-\xi)+L_{II}(-\xi,-\eta) \tag{B74}$$

$$f_I^{(1)}\left(\frac{1}{4}\right)+L_I\left(\frac{1}{4},\frac{d_p+d_m}{2d_m}\right)=\frac{u_0^{(1)}}{2} \tag{B75}$$

$$\int_{d_p/2d_m}^{(d_m+d_p)/2d_m}\left(\frac{E_m}{l}f_I^{(1)\prime}\left(-\frac{1}{4}\right)+N_I\left(-\frac{1}{4},\eta\right)\right)d\eta=0 \tag{B76}$$

$$\int_{-1/4}^{1/4}\left(\frac{E_m}{2l}\left(\frac{d_p+d_m}{2d_m}\right)^2f_I^{(1)\prime\prime\prime}(\xi)-\left(\frac{d_p+d_m}{2d_m}\right)\lambda_I^{(1)\prime}(\xi)+\chi_I^{(1)}(\xi)+P_I\left(\xi,\frac{d_p+d_m}{2d_m}\right)\right)d\xi=0. \tag{B77}$$

According to Eq. (B74) we have $C_1^{(1)}=0$, $C_3^{(1)}=C_4^{(1)}$, therefore the original five unknown coefficients can be reduced into three unknown coefficients $C^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}=C_4^{(1)}$. Then we have simplified expression of Eq. (B59) as

$$f_I^{(1)}(\xi) = C_2^{(1)}\xi + 2C_3^{(1)} ch\gamma\xi - \frac{AB\gamma^2}{48d_m^3}\left(\frac{3}{2}ch\gamma\xi - \frac{1}{2}sh\gamma\xi - 2\gamma\xi sh\gamma\xi\right). \tag{B78}$$

According to Eq. (B75), Eq. (B76) and Eq. (B78), together with Eqs. (B18) and (B20), we can obtain the unknown coefficients $C_2^{(1)}$ and $C_3^{(1)}$ as below,

$$C_2^{(1)} = \frac{2\gamma sh\frac{\gamma}{4}}{4ch\frac{\gamma}{4} + \gamma sh\frac{\gamma}{4}} u_0^{(1)} + \frac{A\gamma^3}{24d_m^3\left(4ch\frac{\gamma}{4} + \gamma sh\frac{\gamma}{4}\right)} \times \left(\begin{array}{l} B\left(2sh\frac{\gamma}{2} - ch\frac{\gamma}{2} + \gamma\right) + 16d_m^2\left(d_m + 3d_p\right)sh\frac{\gamma}{2} \\ -12\left(2+\upsilon_m\right)d_m\left(d_m^2 + 2d_m d_p\right)sh\frac{\gamma}{2} \end{array}\right) \tag{B79}$$

$$C_3^{(1)} = C_4^{(1)} = \frac{u_0^{(1)}}{4ch\frac{\gamma}{4} + \gamma sh\frac{\gamma}{4}} + \frac{-A\gamma^2}{192d_m^3\left(4ch\frac{\gamma}{4} + \gamma sh\frac{\gamma}{4}\right)} \times \left(\begin{array}{l} 24\gamma\left(2+\upsilon_m\right)d_m d_p^2 sh\frac{\gamma}{4} + B\gamma^2 ch\frac{\gamma}{4} - B\gamma ch\frac{\gamma}{4} + 5B\gamma sh\frac{\gamma}{4} - 12Bch\frac{\gamma}{4} \\ +4Bsh\frac{\gamma}{4} + 32\gamma d_m^2\left(d_m + 3d_p\right)sh\frac{\gamma}{4} + 96\left(2+\upsilon_m\right)d_m\left(d_m + d_p\right)^2 ch\frac{\gamma}{4} \end{array}\right). \tag{B80}$$

The unknown coefficients $C^{(1)}$ can be obtained through Eq. (B77). By substituting Eq. (B78), Eq. (B63), Eq. (B66) and Eq. (B22) into Eq. (B77), we can get the analytical solution of coefficient $C^{(1)}$ as below

$$C^{(1)} = \frac{AB\gamma^2}{96d_m^3}\left(\frac{d_m + 2d_p}{2d_m}\right)\left(1-\upsilon_p\right)sh\frac{\gamma}{4} - \frac{\left(\upsilon_m d_m + \upsilon_p d_p\right)}{2d_m}C_2^{(1)} \tag{B81}$$

Finally, basing on the initial solution and the first order perturbation correction of the perturbation solution of governing equations, we can obtain the analytical expressions of the improved approximate perturbation solution of the displacement and stress fields in mineral and protein as below,

$$u_i^m = u_i^{m(0)} + \in_d^2 u_i^{m(1)} \tag{B82}$$

$$v_i^m = \in_d \left(v_i^{m(0)} + \in_d^2 v_i^{m(1)}\right) \tag{B83}$$

$$\sigma_{xx,i}^m = \sigma_{xx,i}^{m(0)} + \in_d^2 \sigma_{xx,i}^{m(1)} \tag{B84}$$

$$\sigma_{yy,i}^m = \in_d^2 \left(\sigma_{yy,i}^{m(0)} + \in_d^2 \sigma_{yy,i}^{m(1)}\right) \tag{B85}$$

$$\sigma_{xy,i}^m = \in_d \left(\sigma_{xy,i}^{m(0)} + \in_d^2 \sigma_{xy,i}^{m(1)}\right) \tag{B86}$$

$$u^p = u^{p(0)} + \in_d^2 u^{p(1)} \tag{B87}$$

$$v^p = \in_d \left(v^{p(0)} + \in_d^2 v^{p(1)}\right) \tag{B88}$$

$$\sigma_{xx}^{p} = \in_d^2 \left(\sigma_{xx}^{p(0)} + \in_d^2 \ \sigma_{xx}^{p(1)}\right) \tag{B89}$$

$$\sigma_{yy}^{p} = \in_d^2 \left(\sigma_{yy}^{p(0)} + \in_d^2 \ \sigma_{yy}^{p(1)}\right) \tag{B90}$$

$$\sigma_{xy}^{p} = \in_d \left(\sigma_{xy}^{p(0)} + \in_d^2 \ \sigma_{xy}^{p(1)}\right) \tag{B91}$$

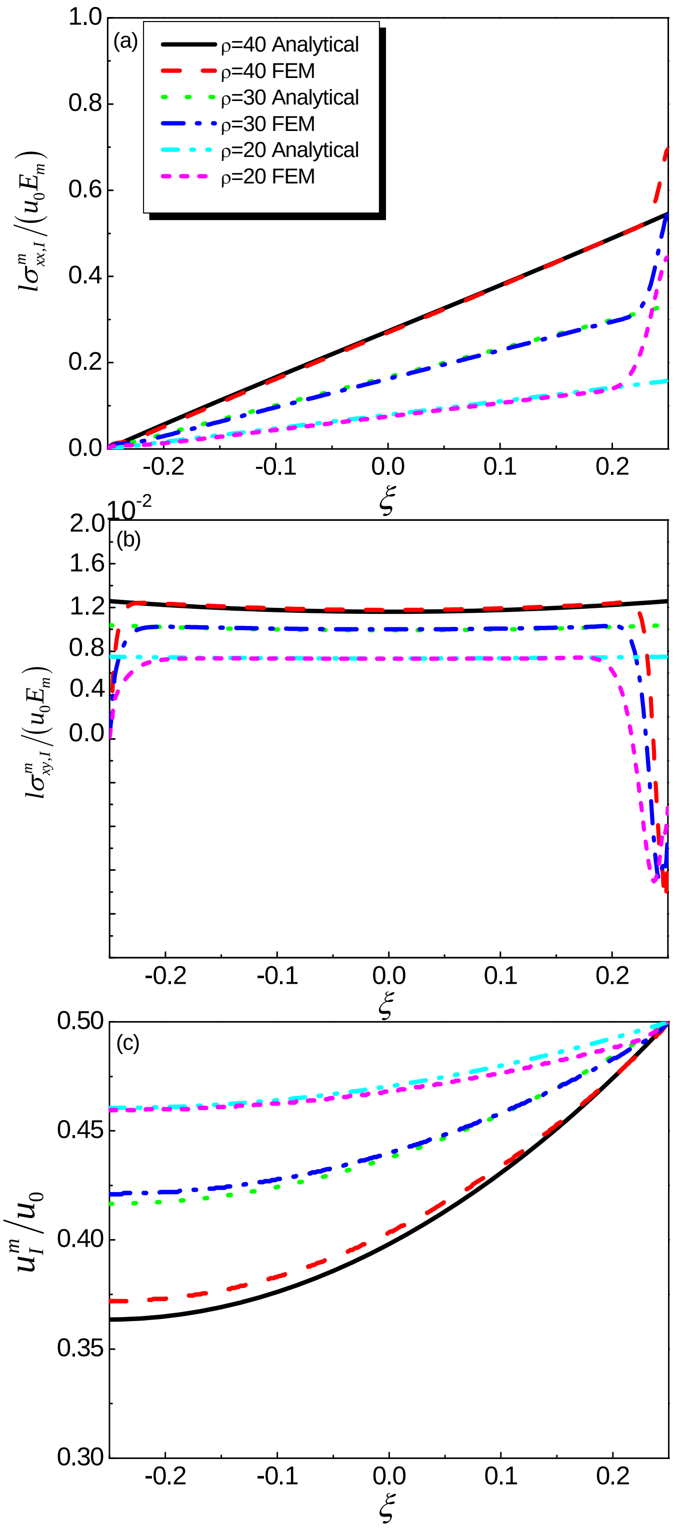

(a)
ρ=40 Analytical
ρ=40 FEM
ρ=30 Analytical
ρ=30 FEM
ρ=20 Analytical
ρ=20 FEM
$l\sigma^m_{xx,I}/(u_0E_m)$
ξ
(b)
10^{-2}
$l\sigma^m_{xy,I}/(u_0E_m)$
ξ
(c)
u^m_I/u_0
ξ

Figure S1

Figure S1: The comparison between our theoretical predictions and the FEM simulation results. (a) The dimensionless normal stress in the mineral crystal; (b) the dimensionless shear stress along the mineral-protein interface; (c) the dimensionless displacement in the mineral along the longitudinal direction. $E_m/G_p = 2400$, and $\Phi = 50\%$.

(Discussion: One reason for the discrepancy between our theoretical predictions and the FEM results at the mineral tips is that we assumed a traction free boundary condition at the mineral tips by neglecting the traction force acted by protein at the mineral tips during the model simplification from Fig. 1(b) to Fig. 2. The other reason is that we neglected the boundary condition at the end of protein layer. We note that there is non-zero shear stress at the end of protein layer, which violates the Newton's third law for the mutual forces of action and reaction between the protein layer and its neighbors at lines *CE* and *DF*. This problem can be solved by constructing a boundary-layer type solution to satisfy the Newton's third law. However, according to theSaint-Venant's principle, these simplifications will only affect the tip zone of the mineral of size about $2\left(d_m + d_p\right)$. If the aspect ratio of the mineral is large, the ratio of this zone size to the length of mineral will be small. The consistency between the theoretical predictions and the FEM results just validated our assumptions)

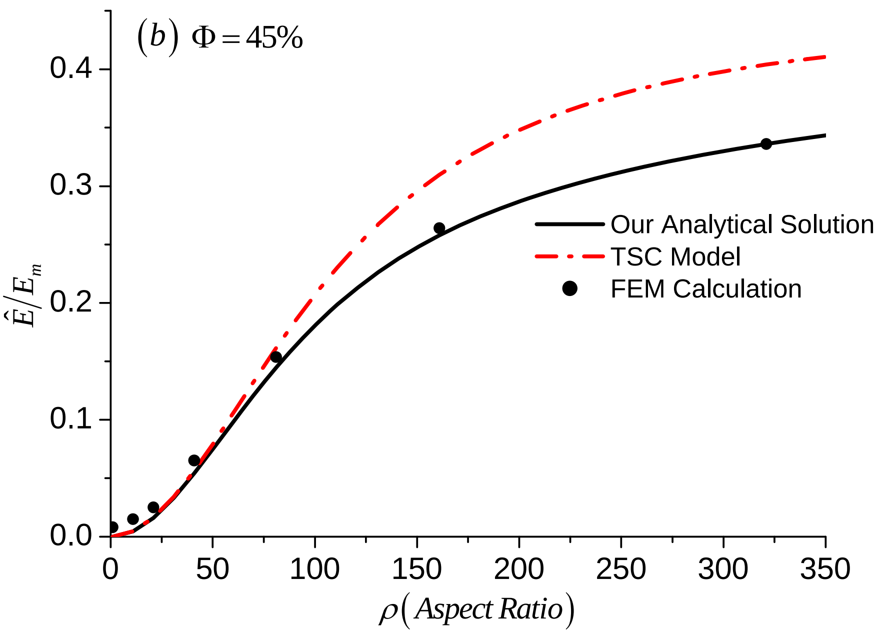
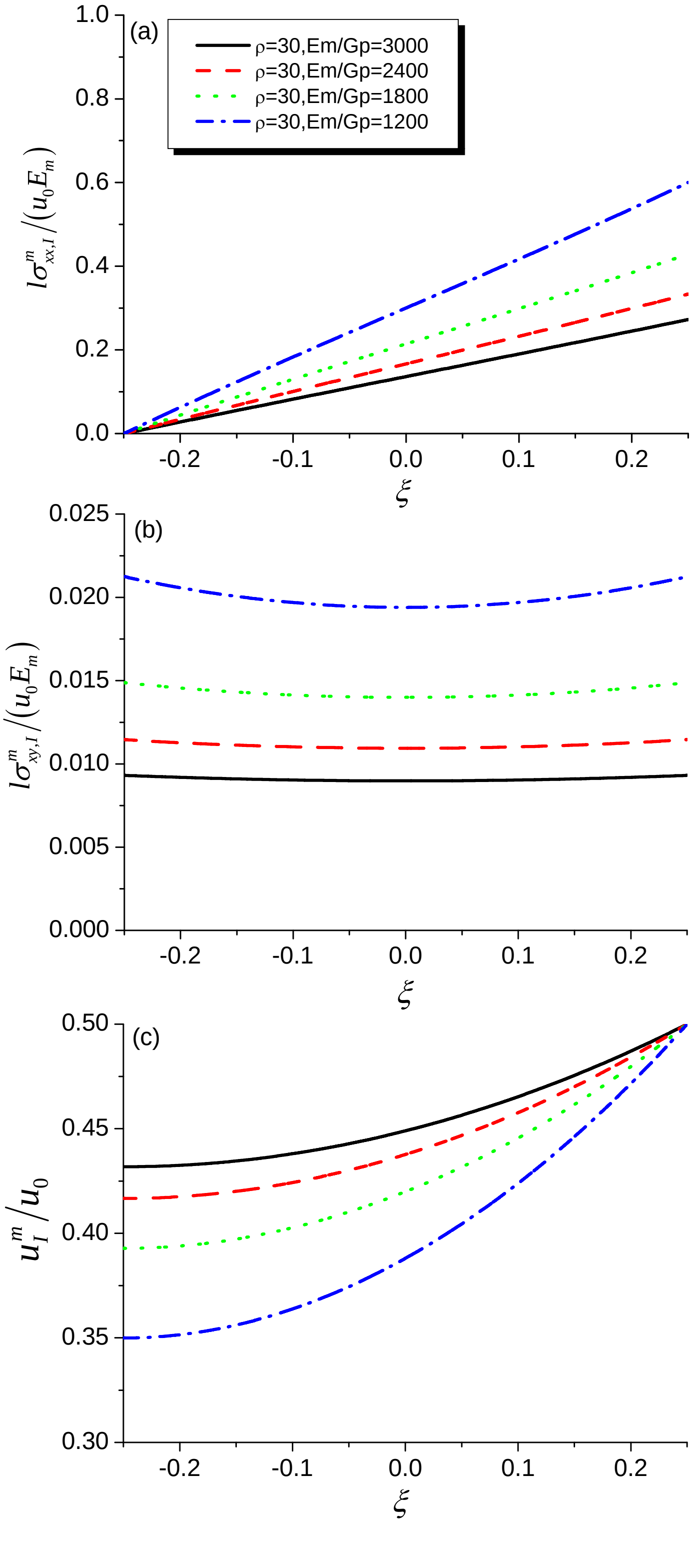


Figure S2

Figure S2: The equivalent Young's modulus of the nanocomposite structure versus the aspect ratio of mineral crystal predicted by our analytical solution, the TSC model and the FEM simulation ($E_m/G_p = 2400$). The volume fraction of mineral $\Phi = 45\%$ (as the case of bone).



(a)
ρ=30,Em/Gp=3000
ρ=30,Em/Gp=2400
ρ=30,Em/Gp=1800
ρ=30,Em/Gp=1200
$l\sigma^{m}_{xx,I}/(u_0E_m)$
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.1
0.0
0.1
0.2
ξ
(b)
$l\sigma^{m}_{xy,I}/(u_0E_m)$
0.025
0.020
0.015
0.010
0.005
0.000
-0.2
-0.1
0.0
0.1
0.2
ξ
(c)
u^{m}_{I}/u_0
0.50
0.45
0.40
0.35
0.30
-0.2
-0.1
0.0
0.1
0.2
ξ

Figure S3

Figure S3: The influence of the elastic modulus ratio of mineral to protein on the distribution of the dimensionless normal stress in mineral, the dimensionless shear stress at the mineral-protein interface and the dimensionless displacement in mineral. $\Phi = 50\%$.

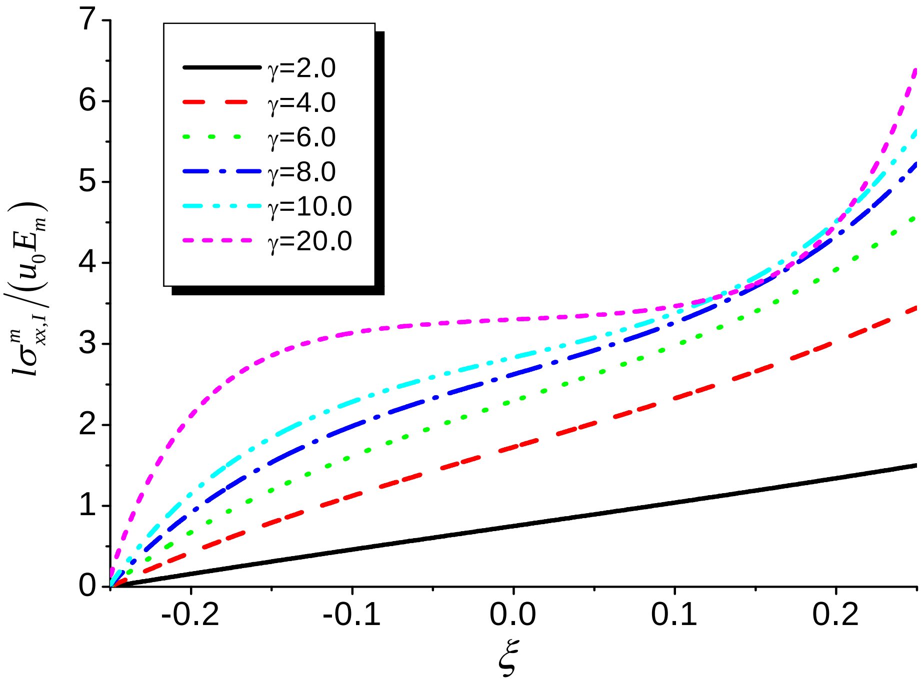


Figure S4

Figure S4: The evolution of the distribution of the normal stress in the mineral along the longitudinal direction with the variation of the γ value.