A STATE OF STRESS IS A LINEAR MAP

A state of stress in a body. Components of stress. Subject to a load, a body develops internal forces. The distribution of the internal forces in the body is usually *inhomogeneous*. For example, when a rod is bent, part of the rod is in tension, and the other part of the rod is in compression.

If the internal forces are uniformly distributed in the body, the body is said to be in a *homogeneous state of stress*. For brevity, we often say a state of stress, or just stress.

It has been a common practice to define a state of stress by defining its components. Here is the procedure. Draw a free-body diagram using a rectangular part of the body. Represent the internal forces in the body by a force on each face of the rectangular block. The force is a vector of three components, one being normal to the face, and the other two being tangential to the face. A component of the force divided by the area of the face defines a *component of stress*.

One state of stress, many sets of components. A state of stress in the body is a fact. A choice of block in our mind is an artifact. For a body in a homogeneous state of stress, the components of stress are independent of the location, size and shape of the rectangular block. But the components of stress do depend on the orientation of the block. The state of stress is, of course, independent of the choice of block. Here are the key points:

- One state of stress in a body
- Many blocks in different orientations
- Many sets of components of stress on the faces of various blocks

We commonly resolve this issue by relating the components of stress on the faces of a block in one orientation to the components of stress on the faces of a block in another orientation.

Force, Area, Stress

But why should we even bother with any block at all? We now define a state of stress without using any block. To do so requires us to invoke the fundamental ideas in linear algebra: vector space, and linear map between vector spaces.

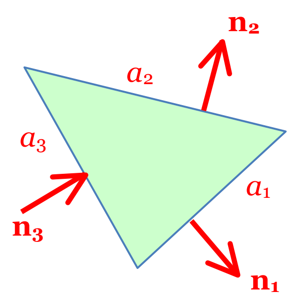
These notes assume that you have learned linear algebra. If vector space and linear map sound vaguely familiar to you, read on until they make you really uncomfortable. These mathematical concepts have formal definitions. Look them up online, starting with Wikipedia.

The set of force form a vector space. The set of forces form a vector space. A force $\mathbf{f}$ times a real number α is another force, written as $\alpha\mathbf{f}$. The sum of two forces, $\mathbf{f}_1 + \mathbf{f}_2$, is another force. The sum of two forces follows the rule of parallelogram.

The set of planar regions form a vector space. The object $\mathbf{a} = a\mathbf{n}$ represents a planar region of area a, normal to a unit vector $\mathbf{n}$. We next confirm that the set of planar regions form a vector space.

The object $-\mathbf{a}$ represents a planar region of area a, also normal to the unit vector $\mathbf{n}$. Let β be a positive number. Thus, the object $\beta\mathbf{a}$ represents a planar region of area βa normal to the unit vector $\mathbf{n}$. Taken together, we have confirmed that, for every planar region $\mathbf{a}$ and every number β, the product $\beta\mathbf{a}$ is also a planar region.

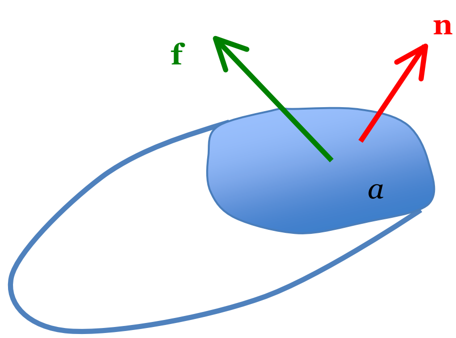
Consider two planar regions represented by $\mathbf{a}_1$ and $\mathbf{a}_2$. Because the shapes of the planar regions do not affect the definition of the stress, we may choose the two regions as rectangular regions. The sum $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ represents another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is shown in the figure. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism.



A state of stress maps a planar region to the force acting on the planar region. A body is in a homogeneous state of stress. Inside the body, consider a planar region of area a, normal to a unit vector $\mathbf{n}$. Because the body is in a homogeneous state of stress, acting on the planar region is a uniformly distributed force. We represent this uniformly distributed force by resultant force $\mathbf{f}$ acting on the centroid of the planar region. The force depends on both the area and direction of the planar region. We write this relation as a function:

$$\mathbf{f} = \mathbf{T}(\mathbf{a}).$$

The input of the function $\mathbf{T}$ is a vector representing a planar region, and the output of the function another vector representing the force acting on the planar region. The function $\mathbf{T}$ defines a state of stress. Thus, a state of stress maps a planar region to the force acting on the region.



The balance of forces requires that the a state of stress be a linear map. In linear algebra, a function that maps one vector space to another vector space is a linear map if

1. $\mathbf{T}(\beta\mathbf{a}) = \beta\mathbf{T}(\mathbf{a})$ for every number β and every vector $\mathbf{a}$, and
2. $\mathbf{T}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{T}(\mathbf{a}_1) + \mathbf{T}(\mathbf{a}_2)$ for any two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

We next prove that the function $\mathbf{f} = \mathbf{T}(\mathbf{a})$ is indeed a linear map.

Let β be a positive number. Because the body is in a homogeneous state of stress, the internal forces in the body are uniformly distributed. In particular, the force acting on the planer region $\beta\mathbf{a}$ is linear in β:

$$\mathbf{T}(\beta\mathbf{a}) = \beta\mathbf{T}(\mathbf{a}).$$

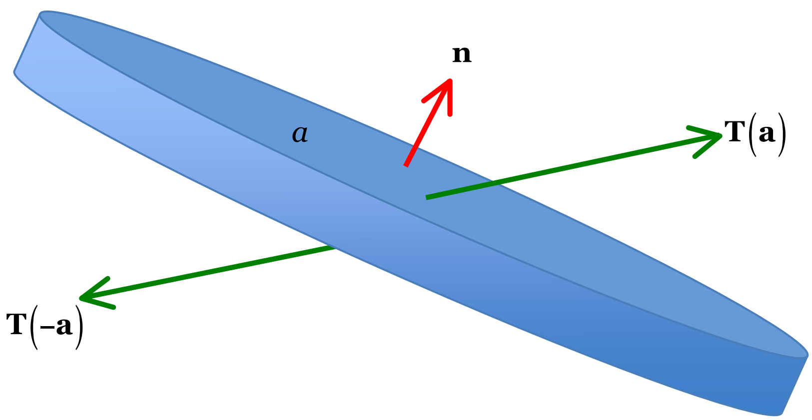
Consider a thin slice of the material. Let $\mathbf{a}$ be one face of the slice, and $-\mathbf{a}$ be the other face of the slice. In each case, the unit vector normal to the face points outside the slice. The forces acting on the two faces are $\mathbf{T}(\mathbf{a})$ and $\mathbf{T}(-\mathbf{a})$. The balance of forces acting on the slice requires that

$$\mathbf{T}(-\mathbf{a}) = -\mathbf{T}(\mathbf{a}).$$

The combination of the above statements shows that the function obeys

$$\mathbf{T}(\beta\mathbf{a}) = \beta\mathbf{T}(\mathbf{a})$$

for every area vector $\mathbf{a}$ and every number β.

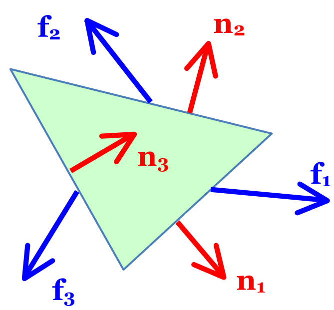


Consider two planar regions $\mathbf{a}_1$ and $\mathbf{a}_2$. Once again, because the shapes of the two regions do not affect the definition of the stress, we choose the two regions as rectangular regions. The sum $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ is another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is shown in the figure. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism.

The forces acting on the three faces of the prism are $\mathbf{f}_1 = \mathbf{T}(\mathbf{a}_1)$, $\mathbf{f}_2 = \mathbf{T}(\mathbf{a}_2)$ and $\mathbf{f}_3 = \mathbf{T}(-\mathbf{a}_3)$. The prism is a free-body diagram. The forces acting on the three faces are balanced, $\mathbf{f}_3 + \mathbf{f}_1 + \mathbf{f}_2 = \mathbf{o}$, so that

$$\mathbf{T}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{T}(\mathbf{a}_1) + \mathbf{T}(\mathbf{a}_2).$$

This equation holds for any planar regions.



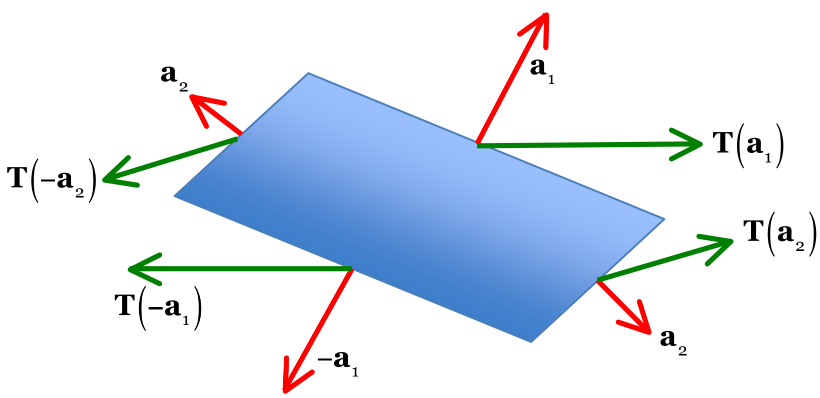
We have confirmed that the function $\mathbf{f} = \mathbf{T}(\mathbf{a})$ is a *linear map* that maps the vector space of planar regions to the vector space of forces. In linear algebra, a linear map from one vector space to another vector space is also called a *tensor*. In mechanics, we call this particular linear map a *state of stress*. That is, a state of stress is a linear map that maps a planar region to the force acting on the planar region.

The balance of moments requires that a state of stress be a symmetric linear map. In linear algebra, a linear map is said to be symmetric if

$$\mathbf{a}_1 \cdot \mathbf{T}(\mathbf{a}_2) = \mathbf{a}_2 \cdot \mathbf{T}(\mathbf{a}_1)$$

for any two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

Consider two planar regions represented by $\mathbf{a}_1$ and $\mathbf{a}_2$. As before we choose the two regions as rectangular regions. Let $-\mathbf{a}_1$ be a planar region in parallel with the planar region $\mathbf{a}_1$, and $-\mathbf{a}_2$ be a planar region in parallel with the planar region $\mathbf{a}_2$. The four planar regions form the faces of a prism. The cross section of the prism is a parallelogram shown in the figure. We choose the length L of the prism (not shown in the figure) to be much larger than the transverse size of the prism. Acting on the faces are the forces $\mathbf{T}(\mathbf{a}_1)$, $\mathbf{T}(\mathbf{a}_2)$, $\mathbf{T}(-\mathbf{a}_1)$ and $\mathbf{T}(-\mathbf{a}_2)$.



Because the internal forces are uniformly distributed, each force acts at the center of a face. The balance of forces requires that $\mathbf{T}(-\mathbf{a}_1) = -\mathbf{T}(\mathbf{a}_1)$ and $\mathbf{T}(-\mathbf{a}_2) = -\mathbf{T}(\mathbf{a}_2)$. Consequent, the two forces $\mathbf{T}(\mathbf{a}_1)$ and $\mathbf{T}(-\mathbf{a}_1)$ form a couple, and the two forces $\mathbf{T}(\mathbf{a}_2)$ and $\mathbf{T}(-\mathbf{a}_2)$ form another couple. The moment of the couple $\mathbf{T}(\mathbf{a}_1)$ and $\mathbf{T}(-\mathbf{a}_1)$ is $(\mathbf{a}_2 / L) \cdot \mathbf{T}(\mathbf{a}_1)$. Similarly, the moment of the couple $\mathbf{T}(\mathbf{a}_2)$ and $\mathbf{T}(-\mathbf{a}_2)$ is $(\mathbf{a}_1 / L) \cdot \mathbf{T}(\mathbf{a}_2)$. The balance of moments acting on the prism requires that

$$\left(\frac{\mathbf{a}_1}{L}\right) \cdot \mathbf{T}(\mathbf{a}_2) = \left(\frac{\mathbf{a}_2}{L}\right) \cdot \mathbf{T}(\mathbf{a}_1).$$

That is, the state of stress $\mathbf{T}$ is a symmetric linear map.

Components Relative to a Basis

Basis. The preceding development is independent of the choice of the basis in the vector space. We next choose a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in the vector space. A vector space equipped with inner product is known as a Euclidean space. The base vectors are ordered to follow the right-hand rule. Each base vector is a unit vector:

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1 .$$

Two different base vectors are normal to each other:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0 .$$

This basis is known as an *orthonormal basis*. The orthonormal base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are edges of a unit cube.

Components of a force relative to a basis. Once we choose a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the force $\mathbf{f}$ is a linear combination of the base vectors:

$$\mathbf{f} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3 ,$$

where f_1, f_2, f_3 are the components of force $\mathbf{f}$ relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The force $\mathbf{f}$ is independent of the choice of basis, but the components of the force depend on the choice of basis.

We often write the above equation in shorthand:

$$\mathbf{f} = f_i \mathbf{e}_i .$$

This way of writing follows a convention: a repeated index implies summation over 1,2,3. (This is called the Einstein summation convention.) Because the sum is the same whatever the repeated index is named, such an index is called a dummy index summation is implied over the repeated indices.

We also often write the components of a force by a column:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Components of a planar region relative to a basis. The above considerations apply to any vector. For example, a unit vector $\mathbf{n}$ is a linear combination of the base vectors:

$$\mathbf{n} = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + n_3 \mathbf{e}_3 ,$$

where n_1, n_2, n_3 are the components of the unit vector $\mathbf{n}$ relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

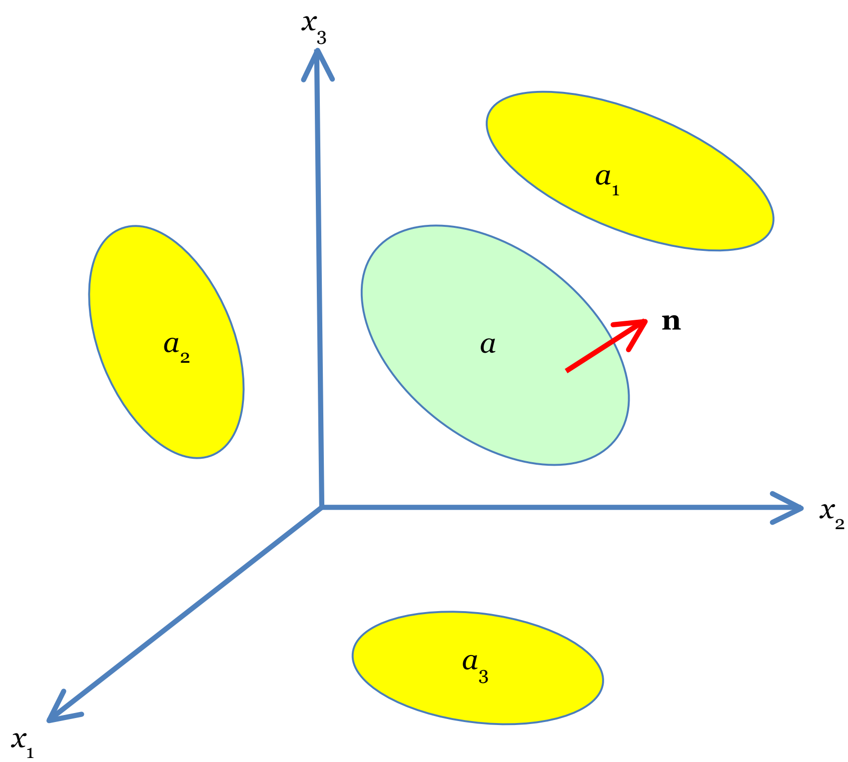
Consider a planar region of area a normal to a unit vector $\mathbf{n}$. We write the planar region as a vector $\mathbf{a} = a\mathbf{n}$. The planar-region vector $\mathbf{a}$ is also a linear combination of the base vectors:

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 ,$$

where a_1, a_2, a_3 are the components of the planar-region vector $\mathbf{a}$ relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Note the relations:

$$a_1 = an_1, \quad a_2 = an_2, \quad a_3 = an_3 .$$

These algebraic relations have a geometric interpretation. The component a_1 is the area of the planar region **a** projected on the coordinate plane normal to $\mathbf{e}_1$. The same are true for the other two components a_2 and a_3.



Components of a state of stress relative to a basis. Acting on the face $\mathbf{e}_1$ of the unit cube is the force $\mathbf{T}(\mathbf{e}_1)$. This force is a vector, which is also a linear combination of the three base vectors:

$$\mathbf{T}(\mathbf{e}_1) = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3,$$

where T_{i1} are the three components of the force relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Similarly, we write

$$\mathbf{T}(\mathbf{e}_2) = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3,$$

$$\mathbf{T}(\mathbf{e}_3) = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3.$$

The force acting on the unit cube on the face whose normal is $-\mathbf{e}_1$ is given by $\mathbf{T}(-\mathbf{e}_1) = -\mathbf{T}(\mathbf{e}_1)$. This algebra is consistent with a physical requirement: the balance of the forces acting on the unit requires that the two forces acting on each pair of parallel faces of the unit cube be equal in magnitude and opposite in direction.

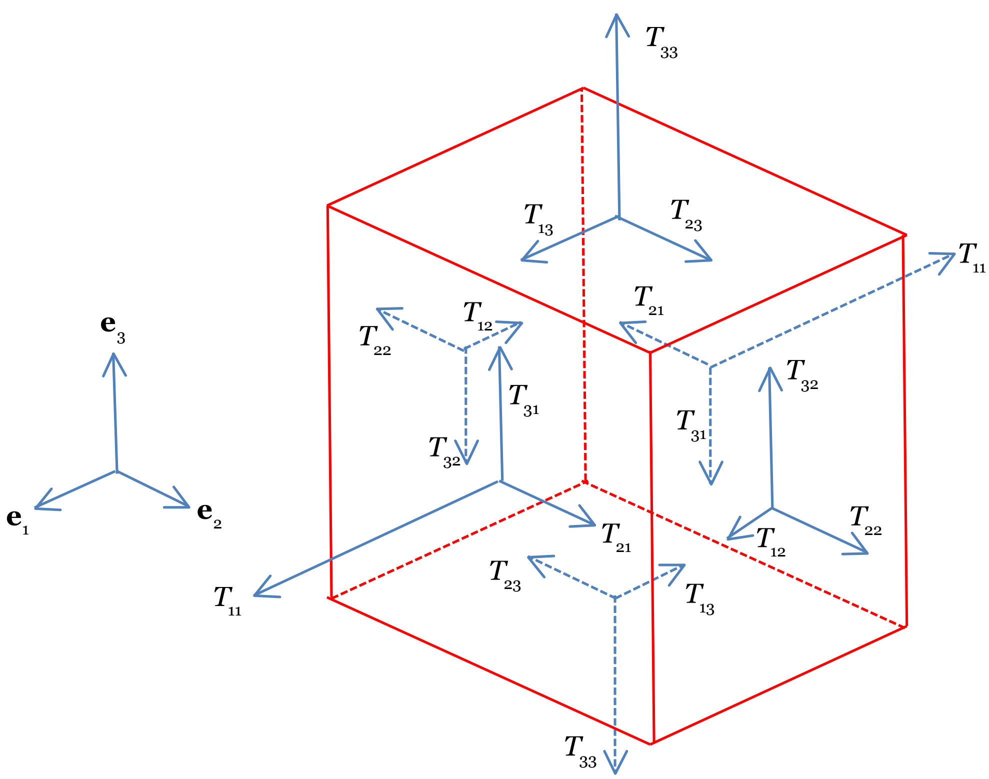
The nine quantities T_{ij} are the components of stress. Using the summation convention, we write the above three expressions as

$$\mathbf{T}\left(\mathbf{e}_j\right) = T_{ij}\mathbf{e}_i .$$

In this equation, i is a dummy index, but j is a free index. This equation represents three independent equations listed above. We list the components of stress as a matrix:

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

The balance of moments requires that the matrix be symmetric, $T_{ij} = T_{ji}$.



Consider a unit cube in the orientation of the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Acting on each face of the cube are three components of stress, one being normal to the face (the normal stress), and the other two being tangential to the face (the shearing stresses). Now the block has six faces, so there are a total of 18 components of stress. Not all the 18 components are independent. The balance of the forces acting on the block requires that the two components of forces acting on each pair of parallel faces of the block be equal in magnitude and opposite in direction. The balance of moments relates the shearing forces acting on four faces around each axis. Consequently, a total of six independent components fully specify a state of stress: three normal stresses along the three axes of the coordinates, and three shearing stresses around the three axis of the coordinates.

The nine quantities T_{ij} are components of forces acting on the faces of the unit cube. The first index indicates the direction of the force, and the second index indicates the direction of the normal vector of the face. When the outward normal vector of the face points in the positive direction of axis j, $T_{ij} > 0$ is positive if the component i of the force points in the positive direction of axis i. When the outward normal vector of the face points in the negative direction of the axis j, $T_{ij} > 0$ if the component i of the force points in the negative direction of axis i.

Relation between force, stress and planar region. Recall the relation

$$\mathbf{f} = \mathbf{T}(\mathbf{a}).$$

We now this relation in terms of components relative to a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Write $\mathbf{f} = f_i \mathbf{e}_i$, $\mathbf{a} = a_j \mathbf{e}_j$ and $\mathbf{T}(\mathbf{e}_j) = T_{ij}\mathbf{e}_i$. Note that

$$\mathbf{T}(\mathbf{a}) = \mathbf{T}(a_j \mathbf{e}_j) = a_j \mathbf{T}(\mathbf{e}_j) = a_j T_{ij} \mathbf{e}_i .$$

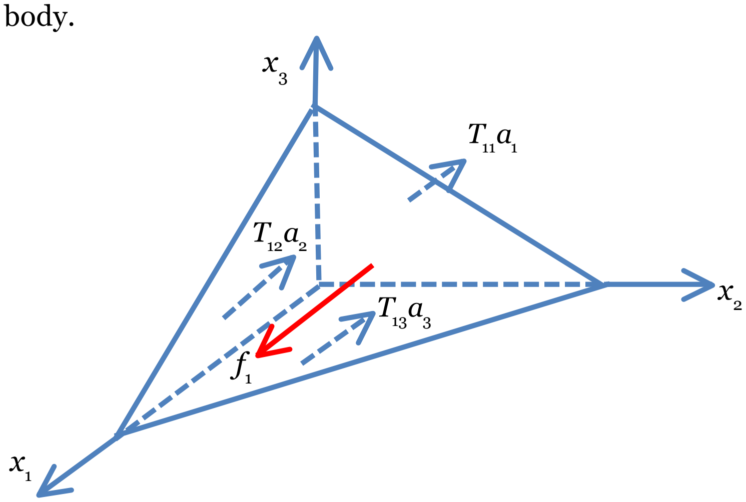
Consequently, the equation $\mathbf{f} = \mathbf{T}(\mathbf{a})$ becomes

$$f_i = T_{ij} a_j .$$

We write the relation using the notation of matrix:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

This relation restates the fact: a state of stress is a linear map that maps every planar region to the force acting on the planar region. We now see the merit of writing the components of a state of stress as a matrix. The six components are indeed sufficient to characterize a state of stress in a body, because the six components allow us to calculate the force on any planar region in the body.



The relation between the components of force, stress and planar region can be confirmed directly by the balance of forces. Consider a tetrahedron formed by a plane and the three coordinate planes. The particular planar region is a triangle. The projections of this planar region on the coordinate planes are three triangles of areas a_1, a_2, a_3. Consider the forces acting on the tetrahedron in the x_1 direction. The force on the particular plane is f_1, and the forces on the three coordinate planes are $T_{11}a_1$, $T_{12}a_2$ and $T_{13}a_3$. The balance of the forces acting on the tetrahedron in the x_1 direction requires that

$$f_1 = T_{11}a_1 + T_{12}a_2 + T_{13}a_3 .$$

The balance of forces in the other two directions leads to similar equations:

$$f_2 = T_{21}a_1 + T_{22}a_2 + T_{23}a_3 ,$$

$$f_3 = T_{31}a_1 + T_{32}a_2 + T_{33}a_3 .$$

These three relations show that a state of stress maps a planar region to a force acting on the planar region.

Transformation of Components due to a Change of Basis

The direction-cosine matrix relating two bases. Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be an orthonormal basis. Let $\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3$ be a new orthonormal basis. Let the angle between the two vectors $\mathbf{e}'_a$ and $\mathbf{e}_i$ be θ_{ai}. That is, $\mathbf{e}'_a \cdot \mathbf{e}_i = \cos\theta_{ai}$. Denote the *direction cosine* of the two vectors by

$$Q_{ai} = \mathbf{e}'_a \cdot \mathbf{e}_i .$$

As a convention, the first index of Q_{ai} indicates a base vector in $\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3$, and the second index indicates a base vector in $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. To help reading, we use the beginning letters in the alphabet for $\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3$, and use the middle letters like *i* and *j* in the alphabet for $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. The first index of Q_{ai} refers to the new basis, and the second to the old basis. For the two bases, there are a total of 9 direction cosines. We can list Q_{ai} as a 3 by 3 matrix. By our convention, the rows refer to the new basis, and the columns to the old basis.

A base vector in the new basis, $\mathbf{e}'_a$, is a linear combination of the base vectors in the old basis,

$$\mathbf{e}'_a = Q_{a1}\mathbf{e}_1 + Q_{a2}\mathbf{e}_2 + Q_{a3}\mathbf{e}_3 .$$

If you are tired of writing sums like this, you abbreviate it by using the summation convention:

$$\mathbf{e}'_a = Q_{ai}\mathbf{e}_i .$$

Similarly, a base vector in the old basis, $\mathbf{e}_i$, is a linear combination of the base vectors in the new basis,

$$\mathbf{e}_i = Q_{ai}\mathbf{e}'_a .$$

The two expressions together give that $\mathbf{e}'_a = Q_{ai}Q_{bi}\mathbf{e}'_b$. Any vector is a unique linear combination of the base vectors, so that

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I} ,$$

where **I** is the identity matrix. That is, the direction-cosine matrix **Q** is an *orthogonal matrix*.

Transformation of components of a vector due to change of basis. Let **f** be a vector. It is a linear combination of the base vectors:

$$\mathbf{f} = f_i\mathbf{e}_i ,$$

where f_1, f_2, f_3 are the components of the vector, and are commonly written as a column. Consider the vector pointing from Cambridge to Boston. When the basis is changed, the vector between Cambridge and Boston remains unchanged, but the components of the vector do change. Let f_1', f_2', f_3' be the components of the vector **f** in the new basis, namely,

$$\mathbf{f} = f_a' \mathbf{e}_a'$$

Recall the transformation between the two bases, $\mathbf{e}_i = Q_{ai}\mathbf{e}_a'$, we write that

$$\mathbf{f} = f_i \mathbf{e}_i = f_i Q_{ai} \mathbf{e}_a'$$

A comparison between the two expressions gives that

$$f_a' = Q_{ai} f_i .$$

Thus, the component column in the new basis is the direction-cosine matrix times the component column in the old basis:

$$\begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Similarly, we can show that

$$f_i = Q_{ai} f_a'$$

The component column in the old basis is the *transpose* of the direction-cosine matrix times the component column in the old basis.

Transformation of the components of stress due to a change of basis. A body is in a homogeneous state of stress. In the body imagine a unit cube in some orientation. The components of stress are the forces acting on the faces of the cube. The state of stress in the body is a physical fact, and is independent of your choice of the basis (i.e., the orientation of the imaginary cube). However, the components of stress depend on your choice of the basis. Given a state of stress, how do we transform the components of stress when the basis is changed?

Relative to the old basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, the components of stress are σ_{ij}, the components of planar region are $a_j = an_j$, and the components of the force acting on the planar region are f_i. Using the summation convention, we write

$$f_i = T_{ij} a_j .$$

Similarly, $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ in the new basis, denote the components of the stress state by T_{ab}', the components of the planar region are $a_b' = an_b'$, and the components of the force acting on the planar region are f_a'. Using the summation convention, we write

$$f_a' = \sigma_{ab}' a_b' . \tag{a}$$

The force is a vector, so that its components transform as $f_a' = Q_{ai} f_i$. The planar region is a vector, and its components transform $a_j = Q_{bj} a_b'$. These two relations, along with $f_i = T_{ij} a_j$ into the above, give

$$f_a' = Q_{ai} T_{ij} Q_{bj} a_b' . \tag{b}$$

Equations (a) and (b) are valid for any choice of the plane. Consequently, we must require that

$$T'_{ab} = Q_{ai} T_{ij} Q_{bj} .$$

Thus, the stress-component matrix in the new basis is the product of three matrices: the the direction-cosine matrix, the stress-component matrix in the old basis, and the transpose of the direction-cosine matrix.

Scalars, vectors, and tensors. When the basis is changed, a scalar (e.g., temperature, energy, and mass) does not change, the components of a vector transform as

$$f'_a = Q_{ai} f_i ,$$

and the components of a tensor transform as

$$T'_{ab} = Q_{ai} T_{ij} Q_{bj} .$$

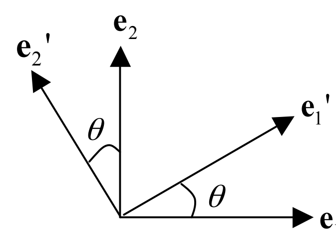
This transformation defines the second-rank tensor. By analogy, a vector is a first-rank tensor, and a scalar is a zeroth-rank tensor. We can also similarly define tensors of higher ranks.

Example. The new basis and the old basis differ by an angle θ around the axis $\mathbf{e}_3$. The sign convention for θ follows the right-hand rule. The direction cosines are

$$\mathbf{e}_1 \cdot \mathbf{e}'_1 = \cos\theta, \quad \mathbf{e}_1 \cdot \mathbf{e}'_2 = -\sin\theta, \quad \mathbf{e}_1 \cdot \mathbf{e}'_3 = 0,$$
$$\mathbf{e}_2 \cdot \mathbf{e}'_1 = \sin\theta, \quad \mathbf{e}_2 \cdot \mathbf{e}'_2 = \cos\theta, \quad \mathbf{e}_2 \cdot \mathbf{e}'_3 = 0,$$
$$\mathbf{e}_3 \cdot \mathbf{e}'_1 = 0, \quad \mathbf{e}_3 \cdot \mathbf{e}'_2 = 0, \quad \mathbf{e}_3 \cdot \mathbf{e}'_3 = 1.$$

Consequently, the matrix of the direction cosines is

$$\left[Q_{ai}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



The components of a vector transform as

$$\begin{bmatrix} f'_1 \\ f'_2 \\ f'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Thus,

$$f'_1 = f_1 \cos\theta + f_2 \sin\theta$$
$$f'_2 = -f_1 \sin\theta + f_2 \cos\theta$$
$$f'_3 = f_3$$

The components of a state of stress transform as

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$T'_{11} = \frac{T_{11}+T_{22}}{2} + \frac{T_{11}-T_{22}}{2}\cos 2\theta + T_{12}\sin 2\theta$$
$$T'_{22} = \frac{T_{11}+T_{22}}{2} - \frac{T_{11}-T_{22}}{2}\cos 2\theta - T_{12}\sin 2\theta$$
$$T'_{12} = -\frac{T_{11}-T_{22}}{2}\sin 2\theta + T_{12}\cos 2\theta$$
$$T'_{13} = T_{13}\cos\theta + T_{23}\sin\theta$$
$$T'_{23} = -T_{13}\sin\theta + T_{23}\cos\theta$$
$$T'_{33} = T_{33}$$

The in-plane components of stress $\left(T_{11}, T_{22}, T_{12}\right)$ transform in the same way as the components of the plane stress. The two out-of-plane shearing stresses $\left(T_{13}, T_{23}\right)$ transform in the same way as components of a vector. The out-of-plane normal stress T_{33} remains unchanged.

Principal Stress

A block in a principal orientation. For any given state of stress, it is always possible to cut a rectangular block in a suitable orientation, such that the components of stress on all faces of the block are normal to the faces, with no shear. These faces are called the *principal planes*, the normal vectors of these faces the *principal directions*, and the stresses on these faces the *principal stresses*. Consider several examples:

- *Uniaxial stress*. One principal direction coincides with the loading axis. The other two principal directions can be any directions transverse to the loading axis**.**
- *Hydrostatic stress*. Any direction is a principal direction.
- *Equal-biaxial stress*. One principal direction is normal to the plane of stress. The other two principal directions can be any directions in the plane of stress.
- *Unequal-biaxial stress*. One principal direction is normal to the plane of stress. The other two principal directions are in the two directions of the applied stresses. In this case, the set of three principal directions is unique.
- *Shearing stress* is the same state of stress as the combination of pulling and pressing in 45°. In this case, the set of three principal directions is unique.

Given a state of stress, how to calculate the principal stresses? Now we return to the principal question. We specify a state of stress by imagining a rectangular block, and then listing the six components of stresses acting on the faces of the block. Given a state of stress in this way, how do we calculate the principal stresses and principal directions?

When a planar region is a principal plane, the force $\mathbf{f}$ acting on the region is in the direction normal to the plane. Write $\mathbf{f} = \sigma \mathbf{a}$, where σ is the magnitude of the force per unit area. Recall that $\mathbf{f} = \mathbf{T}\left(\mathbf{a}\right)$. The above equation becomes

$$\mathbf{T}\left(\mathbf{a}\right) = \sigma \mathbf{a} .$$

This is an eigenvalue problem of the linear map $\mathbf{T}$. We have now answered the principal question. When the state of stress $\mathbf{T}$ is known, solve the above eigenvalue problem to determine the eigenvalue σ and the eigenvector $\mathbf{a}$. The eigenvalue σ is the principal stress, and the eigenvector $\mathbf{a}$ is the principal plane.

The planar-region vector is $\mathbf{a} = a\mathbf{n}$, with the unit vector $\mathbf{n}$ being the direction of the vector, and the area a being the magnitude of the vector. The eigenvalue problem determines the direction $\mathbf{n}$, but not the magnitude a. Rewrite the equation for the eigenvalue problem as

$$\mathbf{T}(\mathbf{n}) = \sigma\mathbf{n} .$$

In the notation of matrix, the above equation becomes

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}.$$

Linear algebra of eigenvalues. Because the stress tensor is a 3 by 3 symmetric matrix, you can always find three real eigenvalues, i.e., principal stresses, $\sigma_a, \sigma_b, \sigma_c$. We distinguish three cases:

(1) If the three principal stresses are unequal, the three principal directions are orthogonal (e.g., shearing stress).
(2) If two principal stresses are equal, but the third is different, the two equal principal stresses can be in any directions in a plane, and the third principal direction is normal to the plane (e.g., uniaxial stress).
(3) If all the three principal stresses are equal, any direction is a principal direction. This state is called a hydrostatic stress.

Maximum normal stress and maximum shearing stress. Let's order the three principal stresses as $\sigma_a \le \sigma_b \le \sigma_c$. This ordering takes into consideration the signs: a compressive stress (negative) is smaller than a tensile stress (positive). On an arbitrary plane, the traction may be decomposed into two components: one component normal the plane (the normal stress), and the other component parallel to the plane (the shearing stress). When you look at a plane with a different normal vector, you find different normal and shearing stresses. You will be delighted by the following theorems:

- Of all planes, the principal plane of σ_c has the maximum normal stress.
- Of all planes, the plane with the normal vector 45° from the principal directions $\mathbf{n}_a$ and $\mathbf{n}_c$ has the maximum shearing stress. The magnitude of the maximum shearing stress is $\tau_{\max} = (\sigma_c - \sigma_a)/2$.

A proofs of the above two theorems are outlined below. Consider a system of coordinates that coincide with three orthogonal directions of the principal stresses, $\sigma_a, \sigma_b, \sigma_c$. Then consider an arbitrary plane whose unit normal vector has components n_1, n_2, n_3 in this coordinate system. The components of the stress tensor in this coordinate system is

$$\begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_b & 0 \\ 0 & 0 & \sigma_c \end{bmatrix}.$$

Thus, on the plane with unit vector (n_1, n_2, n_3), the force per unit area is ($\sigma_a n_1, \sigma_b n_2, \sigma_c n_3$). The normal stress on the plane is

$$\sigma_n = \sigma_a n_1^2 + \sigma_b n_2^2 + \sigma_c n_3^2 .$$

We need to maximize σ_n under the constraint that $n_1^2 + n_2^2 + n_3^2 = 1$.

The shearing stress on the plane τ is given by

$$\tau^2 = (\sigma_a n_1)^2 + (\sigma_b n_2)^2 + (\sigma_c n_3)^2 - (\sigma_a n_1^2 + \sigma_b n_2^2 + \sigma_c n_3^2)^2 .$$

We need to maximize τ under the constraint that $n_1^2 + n_2^2 + n_3^2 = 1$.

Invariants

Invariant of a vector. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis of a Euclidean space. Any vector **u** in the Euclidean space is a linear combination of the base vectors:

$$\mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 .$$

We say that u_1, u_2, u_3 are the components of the vector **u** relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We are familiar with the geometric interpretations of these ideas. The vector **u** is an arrow in the space. The basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ consists of three unit vectors normal to one another. The components u_1, u_2, u_3 are the projection of the vector **u** on to the three unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Once a vector **u** is given in the Euclidean space, the vector itself does not change if we choose another basis. However, the components u_1, u_2, u_3 do change if we choose another basis. We know the rule of the transformation of the components of the same vector relative to two bases.

The sum $u_i u_i$ does not have any free index, and is a scalar. When a new basis is used, the components u_1, u_2, u_3 change, but $u_i u_i$ remains invariant. This invariant has a familiar geometric interpretation: $\sqrt{u_i u_i}$ is the length of the vector **u**. The length of the vector is invariant when the basis changes.

Invariants of a tensor. Let **A** be a second-rank tensor, and A_{ij} be the components of the tensor relative to an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The tensor is symmetric, so that $A_{ij} = A_{ji}$. The components of the tensor form three scalars:

$$A_{ii}, \quad A_{ij} A_{ij}, \quad A_{ij} A_{jk} A_{ki} .$$

We form a scalar by combining the components of the tensor in a way that makes all indices dummy. The three scalars are independent of the choice of the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and are known as the invariants of the tensor **A**.