

BASIC EQUATIONS IN FLUID MECHANICS

Division of labor. The deformation of a body of a fluid is often inhomogeneous. We picture the body as a sum of many small pieces. Each small piece of the fluid (called a fluid element, a material particle, or a material point) consists of a large number of molecules, but is much smaller than the body. The small piece of fluid evolves in time through a sequence of homogeneous states. Different small pieces of fluid communicate with one other to produce the inhomogeneous deformation of the body as a whole. This picture suggests a division of labor into two tasks.

The first task studies homogeneous deformation. We induce homogeneous deformation of a fluid by applying forces and changing temperature. We relate deformation, forces, temperature, etc. by constructing rheological models, also known as constitutive laws, equations of state, or rheological equations of state. Because the homogeneous deformation evolves in time but is invariant in space, the rheological models commonly involve algebraic equations and ordinary differential equations, but not partial differential equations.

The second task studies inhomogeneous deformation. We watch fluids flow outdoors, in laboratories, and on YouTube. We write equations to describe various means of communication among the small pieces of fluid: compatibility of geometry, balance of forces, conservation of mass, conservation of energy, and production of entropy. Because the inhomogeneous deformation evolves in time and varies from piece to piece, these equations of communication are commonly partial differential equations.

The following notes begin with the homogeneous deformation of a small piece, and end with inhomogeneous deformation of a body.

Homogeneous State

We now develop a rheological model of a common fluid undergoing homogeneous deformation. The model describes a rheological state of the fluid using five thermodynamic properties (temperature, pressure, volume, energy, and entropy), and twelve rheological variables (six stresses and six rates of deformation). A rheological state is fully specified by values of two thermodynamic properties and six rheological variables. For instance, we can specify a rheological state by the values of the temperature, pressure, and six rates of deformation. Once we specify the values of two thermodynamic properties, the fluid tells us the values of its three other thermodynamic properties using three thermodynamic equations of state. Once we specify the

values of the six rheological variables, the fluid tells us the values of its six other rheological variables using six rheological equations of state.

This section describes the five thermodynamic properties, twelve rheological variables, three thermodynamic equations of state, and six rheological equations of state.

Name each thermodynamic state of a substance using values of two thermodynamic properties. We will consider a pure fluid, a substance that consists of a large number of molecules of the same species. A fluid can be either a liquid or a gas. When the temperature is low and the pressure is high, the substance is a liquid, and molecules touch one another but frequently change neighbors. When the temperature is high and the pressure is low, the substance is a gas, and molecules fly in space and collide with one another and with the wall of the container; between collisions, on average a molecule flies a distance longer than the size of the molecule.

A substance can be in many thermodynamic states, and has many thermodynamic properties. Examples of thermodynamic properties include temperature, pressure, volume, energy, entropy, smell, color, electric field, and polarization. When the substance is in a particular thermodynamic state, the value of every thermodynamic property is fixed. We name each state of a substance by the values of its properties.

How many properties do we need to differentiate states of a substance? The answer is two. How do we know? The answer depends on the evidence from our experience and on the limit of our attention. One property is too few: if we hold the value of the temperature we can still change the state of the substance by changing the pressure. Three properties are too many: once the temperature and pressure are fixed, the thermodynamic state of the substance is fixed if we neglect minor effects of, say, the electric field and magnetic field.

The choice of two properties, rather than any other number of properties, raises a pedagogical question. Does it make sense to talk about thermodynamics when teaching fluid mechanics? Fluid mechanics can be taught by choosing zero thermodynamic property—that is, by restricting to incompressible fluids and neglecting effects of temperature. This approach avoids thermodynamics altogether, highlights geometry and mechanics, and focuses exclusively on the effects of viscosity and inertia under various initial and boundary conditions. This approach, however, severs the tie between fluid mechanics and thermodynamics, obscures the structure of the theory, and is not general enough to analyze many significant phenomena. Issues and merits on both sides are well recognized.

Here we assume that the thermodynamic state of a fluid is capable of two independent variations. When we specify the values of two properties, the

thermodynamic state of the fluid is fixed, so are all its other thermodynamic properties. Any choice of two properties will do. A popular choice is the temperature and the pressure. Once the two properties are chosen, they serve as coordinates of a plane. A point in the plane represents a thermodynamic state of the substance. Given the values of the two properties, we look up the values of all other properties in thermodynamic tables and diagrams, in textbooks or online.

Thermodynamic equations of state. A thermodynamic equation of state relates thermodynamic properties. In particular, once we have chosen two thermodynamic properties as independent variables, any other thermodynamic property is a function of the two independent properties. For the time being, we consider five thermodynamic properties: temperature, pressure, volume, energy and entropy. We need three thermodynamic equations of state.

Volume, energy, and entropy of a substance are proportional to the amount of the substance. Let v , u and s be the volume, energy, and entropy per unit mass. (The density relates to the volume per unit mass as $\rho = 1/v$.) We have assumed that the thermodynamic state of the substance is capable of two independent variations. Once the volume and energy is specified, the entropy is fixed. This relation is described by a function of two variables:

$$s = s(u, v).$$

This function has been tabulated for many substances. Such a function that expresses one thermodynamic property in terms of two other thermodynamic properties is called a thermodynamic equation of state.

We need two more thermodynamic equations of state to express the other two thermodynamic properties, the temperature T and the pressure p , in terms of the energy and volume. It turns out the two other thermodynamic equations of state relate to the partial derivative of the function $s(u, v)$:

$$\frac{1}{T} = \frac{\partial s(u, v)}{\partial u},$$

$$\frac{p}{T} = \frac{\partial s(u, v)}{\partial v}.$$

The two relations express the temperature and pressure as functions of volume and energy. These two relations, along with the function $s(u, v)$, constitute the three thermodynamic equations of state.

We have just chosen energy and volume as independent variables, and express entropy, temperature and pressure as functions of energy and volume. We can also choose any other two thermodynamic properties as independent variables, and express the other three as functions of these two. For instance, if we choose T and p as independent variables, we can solve u and v from the above two relations, and then use $s(u, v)$ to express s as a function of T and p .

Thermodynamic identity. Recall an identity in calculus:

$$ds = \frac{\partial s(u,v)}{\partial u} du + \frac{\partial s(u,v)}{\partial v} dv .$$

Combining the above expressions, we obtain a thermodynamic identity:

$$ds = \frac{du + pdv}{T} .$$

Thus, the variation in entropy relates to four properties: u , v , p and T . Because these four quantities can be readily measured, this relation is commonly used to determine the function $s(u,v)$ in experimental measurements.

Some additional thermodynamic identities. Starting from the function $s(u,v)$ and the identity $ds = (du + pdv)/T$, we can write many more functions and identities by mathematical manipulations. Of course, such mathematical manipulations will never yield any new information. They do, however, produce identities that are convenient under various experimental conditions.

It is common that s increases with u when v is fixed. Thus, we can invert the function $s(u,v)$ and get another function $u(s,v)$. Rewrite the identity $ds = (du + pdv)/T$ as

$$du = Tds - pdv .$$

Recall an identity in calculus:

$$du = \frac{\partial u(s,v)}{\partial s} ds + \frac{\partial u(s,v)}{\partial v} dv .$$

A comparison of the above two expressions gives that

$$T = \frac{\partial u(s,v)}{\partial s}, \quad p = -\frac{\partial u(s,v)}{\partial v} .$$

These two relations express temperature and pressure as functions of the entropy and volume.

Define the enthalpy by

$$h = u + pv .$$

Rewrite $du = Tds - pdv$ as

$$dh = Tds + vdp .$$

Write

$$T = \frac{\partial h(s,p)}{\partial s}, \quad v = \frac{\partial h(s,p)}{\partial p} .$$

Define the Helmholtz free energy by

$$f = u - Ts .$$

Rewrite $du = Tds - pdv$ as

$$df = -sdT - pdv.$$

Write

$$s = -\frac{\partial f(T, v)}{\partial T}, \quad p = -\frac{\partial f(T, v)}{\partial v}.$$

Define the Gibbs free energy by

$$g = u - Ts + pv.$$

Rewrite $du = Tds - pdv$ as

$$dg = -sdT + vdp.$$

Write

$$s = -\frac{\partial g(T, p)}{\partial T}, \quad v = \frac{\partial g(T, p)}{\partial p}.$$

Newton's law of viscosity. Not all variables in fluid mechanics are thermodynamic properties. Consider a layer of fluid, thickness h and area A , between two rigid plates. Subject to a shear force F , one plate slides relative to the other plate at a velocity V . Newton's law of viscosity says that the velocity V is linear in the force F . In modern words, define shear stress by

$$\tau = F / A.$$

Define shear rate by

$$\gamma = V / h.$$

Write Newton's law of viscosity as

$$\tau = \eta \gamma,$$

where η is the viscosity. We adopt the idealization that the viscosity is independent of the shear rate, so that the shear stress is linear in the shear rate. The viscosity is material specific, and depends on pressure and temperature. Thus, the viscosity is a thermodynamic property.

Neither shear stress nor shear rate is a thermodynamic property, because neither is specified by a thermodynamic state of the substance. Newton's law of viscosity is not a thermodynamic equation of state; rather, it is a rheological equation of state.

We next generalize Newton's law of viscosity by generalizing the stress, the rate of deformation, and the linear relation between them.

Stress. We pull a gum by a force, and define the tensile stress as the force divided by the cross-sectional area of the gum in the deformed state. We now generalize this definition of stress to multiaxial loading.

Consider a fluid in the shape of a unit cube, with faces parallel to the coordinate planes. Now consider one face of the block normal to the axis x_j . Acting on this face is a force. Denote by σ_{ij} the component i of the force acting

on this face.

We adopt the following sign convention. When the outward normal vector of the area points in the positive direction of axis x_j , we take σ_{ij} to be positive if the component i of the force points in the positive direction of axis x_i . When the outward normal vector of the area points in the negative direction of the axis x_j , we take σ_{ij} to be positive if the component i of the force points in the negative direction of axis x_i .

The nine components of stress form a three-by-three matrix. The matrix is symmetric, $\sigma_{ij} = \sigma_{ji}$, as required by the balance of moment acting on the cube.

Stress is a tensor. The matrix σ_{ij} fully describes the state of stress in a piece of material: the matrix allows us to calculate force per unit area on a plane in any direction. Consider a plane of unit normal vector \mathbf{n} . Define the traction \mathbf{t} as force acting on the plane divided by the area of the plane. Consider a tetrahedron formed by the plane and the three coordinate planes. The balance of forces acting on the tetrahedron requires that

$$t_i = \sigma_{ij} n_j.$$

The stress is a linear map that maps one vector (the unit normal vector) to another vector (the traction). In linear algebra, such a linear map is called a tensor.

Rate of extension. Consider two material particles in a fluid moving apart. The distance between the two material particles is a function of time, $l(t)$. Define the rate of extension along the line through the two material particles by

$$D = \frac{dl(t)}{ldt}.$$

That is, the rate of extension is the relative velocity between the two material particles divided by the distance between them.

For a fluid undergoing a homogeneous deformation, the rate of extension depends on time and on the direction of the line. Denote the rates of extension along the three coordinates by D_{11} , D_{22} and D_{33} .

Rate of dilation. A piece of fluid consists of a fixed number of molecules. The volume of the piece is a function of time, $V(t)$. Define the rate of dilation by

$$\frac{dV}{Vdt}.$$

We now relate the rate of dilation to the rates of extension. Consider a simple case that the fluid is of a rectangular shape, sides l_1, l_2, l_3 in the current state. The rates of extension of the sides are

$$D_{11} = \frac{dl_1}{l_1 dt}, \quad D_{22} = \frac{dl_2}{l_2 dt}, \quad D_{33} = \frac{dl_3}{l_3 dt}.$$

The volume of the piece is $V = l_1 l_2 l_3$. The volume changes at the rate

$$\frac{dV}{dt} = \frac{d}{dt}(l_1 l_2 l_3) = l_2 l_3 \frac{dl_1}{dt} + l_3 l_1 \frac{dl_2}{dt} + l_1 l_2 \frac{dl_3}{dt}.$$

The rate of dilation is

$$\frac{dV}{Vdt} = D_{11} + D_{22} + D_{33}.$$

Even though we have derived this result using a fluid of rectangular shape, the conclusion is correct for a fluid of any shape, so long as the deformation is homogeneous. We will show that the three rates of extension are components of a tensor. The sum $D_{11} + D_{22} + D_{33}$ is the trace of the tensor, and is an invariant of the tensor. We write the trace by using the summation convention, D_{kk} .

Rate of shear. Consider three material particles P, A and B in a piece of fluid undergoing homogeneous deformation. At a give time, line PA is normal to line PB, but the two lines rotate relative to each other. We define the rate of shear by one half of the rate at which the angle between the two lines changes. This definition is one half of what we have used in describing the parallel-plate experiment, V/h . Denote the rates of shear with respect to the three coordinates by D_{23} , D_{31} and D_{12} .

Rate of deformation is a tensor. In defining the rates of shear, D_{23} , D_{31} and D_{12} , we have introduced a factor of 2. We do so to make the three rates of shear, along with the three rates of extension, D_{11} , D_{22} and D_{33} , constitute the components of a symmetric, second-rank tensor D_{ij} .

To see that the six rates of deformation are indeed the components of a tensor, we abstract the definition as follows. Consider two points A and B, of coordinates \mathbf{x}^A and \mathbf{x}^B , and velocities \mathbf{v}^A and \mathbf{v}^B . The piece of liquid is

undergoing a homogeneous deformation. That means that the velocity field is linear in space:

$$\mathbf{v}^B - \mathbf{v}^A = \mathbf{L}(\mathbf{x}^B - \mathbf{x}^A).$$

The linear map \mathbf{L} maps one vector $\mathbf{x}^B - \mathbf{x}^A$ to another vector $\mathbf{v}^B - \mathbf{v}^A$. This linear map is called the velocity gradient, and is a tensor. The deformation is homogeneous when the velocity gradient \mathbf{L} is independent of position in space.

Using a picture, we can readily confirm that the rate of deformation relates to the velocity gradient as

$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji}).$$

Thus, the rate of deformation is a tensor.

For a piece of fluid undergoing homogeneous deformation, the tensor D_{ij} describes the rate of extension of a line of material particles in any direction. Consider a line of material particles in the piece. In the current state, the line is in the direction of a unit vector \mathbf{n} , and is of length l . Write $\mathbf{x}^B - \mathbf{x}^A = l\mathbf{n}$. The line of material particles elongates at a rate $dl/dt = \mathbf{n} \cdot (\mathbf{v}^B - \mathbf{v}^A)$. By definition, the rate of extension of the line of material particles is $D_n = (dl/dt)/l$, or $D_n = \mathbf{n} \cdot (\mathbf{L}\mathbf{n})$. This quadratic form is the same as

$$D_n = D_{ij}n_i n_j.$$

The tensor D_{ij} also describes the rate of shear of two lines of material particles. In the current state, the two lines are two unit vectors normal to each other, \mathbf{m} and \mathbf{n} . By definition, the rate of shear with respect to the two vectors is $D_{mn} = (\mathbf{m} \cdot (\mathbf{L}\mathbf{n}) + \mathbf{n} \cdot (\mathbf{L}\mathbf{m}))/2$. This expression is the same as

$$D_{mn} = D_{ij}m_i n_j.$$

Rheological state. The thermodynamic state of a substance is capable of two independent variations. We name each thermodynamic state of the substance by the values of two thermodynamic properties, for example, the temperature and the pressure. Once the temperature and pressure are specified, the values of volume, energy and entropy are fixed.

To these five thermodynamic properties, we have just added six components of stress and six components of rate of deformation. The twelve newcomers are not thermodynamic properties. We now assume that two thermodynamic properties, along with six out of twelve components of stress and rate of deformation, specify a rheological state of the substance.

Rheological equations of state. We now need to relate the six components of the stress to the six components of the rate of deformation by constructing six rheological equations of state. We do so by generalizing Newton's law of viscosity. By definition, $\gamma = 2D_{12}$. We write Newton's law of viscosity as

$$\sigma_{12} = 2\eta D_{12}.$$

We further assume that the fluid is isotropic, so that the same viscosity applies in all directions. We write

$$\sigma_{13} = 2\eta D_{13},$$

$$\sigma_{23} = 2\eta D_{23}.$$

What happens to a fluid in extension or under hydrostatic compression? We next generalize Newton's law of viscosity to any linear, isotropic, viscous fluid.

Generalized Newton's law of viscosity. When all the components of the rate of deformation vanishes, $D_{ij} = 0$, a fluid is in a state of thermodynamic equilibrium, specified by a temperature T and a pressure p . To maintain this state of thermodynamic equilibrium, all the shear stresses must vanish, $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$, and all the normal stresses must be the same as the thermodynamic pressure, $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$. We write the six equations of equilibrium collectively as

$$\sigma_{ij} + p\delta_{ij} = 0.$$

When $\sigma_{ij} + p\delta_{ij} \neq 0$, the fluid is not in thermodynamic equilibrium with the stress, and the rate of deformation does not vanish, $D_{ij} \neq 0$. As a generalization of Newton's law of viscosity, we require that $\sigma_{ij} + p\delta_{ij}$ be linear in D_{ij} . For an isotropic liquid, the linear relation between two symmetric tensors takes the general form:

$$\sigma_{ij} + p\delta_{ij} = 2\eta \left(D_{ij} - \frac{1}{3} D_{kk} \delta_{ij} \right) + \zeta D_{kk} \delta_{ij},$$

where η and ζ depend on two independent thermodynamic properties, but are independent of the rate of deformation. This rheological model provides six equations that connect the two tensors, the stress and the rate of deformation.

This rheological model generalizes Newton's law of viscosity. The model describes linear, isotropic, viscous behavior in its full generality. To see this generalization clearly, we should go through the same process as we obtain the generalized Hooke's law of elasticity. We should consider a fluid under a tensile stress, and look at both the axial extension and transverse contractions. We then relate them to the fluid under shear. We show only two independent material

constants are needed. This process is less elegant on paper, but more intuitive to the brain, than what we have put on paper.

We can also motivate the model in yet another way. The trace of the rate of deformation, D_{kk} , describes the rate at which the piece of liquid changes its volume. The deviatoric part of the rate of deformation, $e_{ij} = D_{ij} - D_{kk}\delta_{ij}/3$, describes the rate at which the piece of liquid changes its shape. The mean stress is $\sigma_m = \sigma_{kk}/3$, and the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}$. The rheological model takes the form

$$s_{ij} = 2\eta e_{ij}, \quad \sigma_m + p = \zeta D_{kk}.$$

The constant η represents the shear viscosity that resists the change in shape, and the constant ζ represents the dilation viscosity that resists the change in volume.

The dilation of a fluid is viscoelastic. Let us examine the pair of equations in the kinetic model separately. Whereas $s_{ij} = 2\eta e_{ij}$ generalizes Newton's law of viscosity, $\sigma_m + p = \zeta D_{kk}$ is a viscoelastic model of the Kelvin type. The thermodynamic pressure related to specific volume and temperature through a thermodynamic equation of state, $p = f(v, T)$. This expression describes the elasticity of the fluid in dilation. Recall the expression for the rate of dilation $D_{kk} = (dv/dt)/v$, we rewrite $\sigma_m + p = \zeta D_{kk}$ as

$$\sigma_m + f(v, T) = \zeta \frac{dv}{v dt}.$$

This expression represents a spring and a dashpot in parallel, subject to an applied stress σ_m .

Note that in general $\sigma_m \neq -p$. When a stress σ_m is suddenly applied and held at a constant, the fluid creeps and the thermodynamic pressure of the fluid approaches this applied stress. The time needed is estimated by ζ/B , where B is bulk elastic modulus of the fluid. Take representative values for water at the room temperature, $\zeta = 10^{-3} \text{ Pa}\cdot\text{s}$ and $B = 10^9 \text{ Pa}$, the relaxation time is $\zeta/B = 10^{-12} \text{ s}$.

For many substances, as the temperature drops, viscosity increases steeply, but the elastic modulus does not. It is conceivable that viscoelastic dilation can be important. It is also conceivable that when viscoelastic dilation is important, viscoelastic shear is also important. To describe shear, viscoelasticity of the Kelvin type is clearly wrong, and we have to invoke viscoelasticity of the Maxwell type, or some form of hybrid. The matter is complex. It is perhaps wise to eliminate the dilation viscosity for fluids like water and air, and deal with viscoelastic fluids in a separate formulation of the theory.

The energy of an isolated system never changes. A unit mass of a fluid is subject to external forces, and is connected to a reservoir of energy. For instance, the external forces can be applied by a set of hanging weights, and the reservoir of energy can be a radiator. We make the fluid, the hanging weights, and the radiator together into an isolated system—it exchanges neither energy nor matter with the rest of the world.

As the fluid deforms at the rate of deformation D_{ij} , the weights change the potential energy at the rate $-\nu\sigma_{ij}D_{ij}$. As the radiator transfers energy to the fluid, the energy of the radiator changes at the rate $-\nu Q$. The energy of the isolated system is the sum of its parts: the fluid, the hanging weights, and the radiator. The energy of the isolated system should never change:

$$\frac{du}{dt} - \nu\sigma_{ij}D_{ij} - \nu Q = 0.$$

This equation expresses the conservation of energy.

The entropy of an isolated system never decreases. The fluid changes entropy at the rate ds/dt . The radiator itself is in a state of thermodynamic equilibrium, held at a constant temperature T_R . When the radiator changes energy at the rate $-\nu Q$, the entropy of the radiator changes at the rate $-\nu Q/T_R$. As the weights move up and down, they do not change entropy. The entropy of the isolated system is the sum of the parts. The law of entropy requires that the entropy of the isolated system should never decrease:

$$\frac{ds}{dt} - \frac{\nu Q}{T_R} \geq 0.$$

The equality holds when the isolated system is in thermodynamic equilibrium, whereas the inequality holds when the isolated system is not in thermodynamic equilibrium.

Thermodynamic condition in terms of independent changes. In the above thermodynamic condition, the change in entropy and the transfer of energy are not independent changes. We now express this condition in terms of seven independent changes. Recall the thermodynamic identity $ds = (du + pdv)/T$ and the geometric relation $dv/dt = \nu D_{kk}$. We obtain that

$$\frac{(\sigma_{ij} + p\delta_{ij})D_{ij}}{T} + \left(\frac{1}{T} - \frac{1}{T_R}\right)Q \geq 0.$$

This thermodynamic condition holds for independent changes, the rate of deformation D_{ij} and rate of transfer Q , a total of seven independent variables.

We satisfy the inequality as follow. We require that the fluid be in thermal equilibrium with the radiator, $T = T_R$. The above inequality reduces to $(\sigma_{ij} + p\delta_{ij})D_{ij} \geq 0$. Note the identity $\sigma_{ij}D_{ij} = s_{ij}e_{ij} + \sigma_m D_{kk}$, and write the inequality as

$$s_{ij}e_{ij} + (\sigma_m + p)D_{kk} \geq 0.$$

We satisfy this inequality by adopting the generalized Newton's law of viscosity, $s_{ij} = 2\eta e_{ij}$ and $\sigma_m + p = \zeta D_{kk}$. The thermodynamic inequality holds when both viscosities η and ζ are nonnegative.

Inhomogeneous State

A body of fluid is a sum of many small pieces. As the body evolves by inhomogeneous deformation, each piece evolves through a sequence of homogeneous states, as described in the previous section. Different pieces communicate through the compatibility of geometry, balance of forces, conservation of mass, and conservation of energy.

Fourier's model of thermal conduction. Let \mathbf{q} be the flux of energy due to thermal conduction. Consider a plane fixed in plane, and let \mathbf{n} be the unit vector normal to the plane. The quantity $\mathbf{q} \cdot \mathbf{n}$ is the amount of energy conducted across the plane per unit area per unit time. Fourier's model requires that the conductive flux of energy should be proportional to the gradient of temperature:

$$q_i = -\kappa \frac{\partial T}{\partial x_i}.$$

The thermal conductivity κ is specific to the substance, and depends on the temperature and pressure. Thus, the thermal conductivity is a thermodynamic property.

Time derivative of a function of material particle. At time t , a material particle \mathbf{X} moves to position $\mathbf{x}(\mathbf{X}, t)$. The velocity of the material particle is

$$\mathbf{V} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}.$$

Here, the independent variables are the time and the coordinates of the material particles when the body is in the reference state.

We can also use \mathbf{x} as the independent variable. To do so we change the variable from \mathbf{X} to \mathbf{x} by using the function $\mathbf{X}(\mathbf{x}, t)$, and then write

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\mathbf{X}, t).$$

Both $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{V}(\mathbf{X}, t)$ represent the same physical object: the velocity of material particle \mathbf{X} at time t . Because of the change of variables, from \mathbf{X} to \mathbf{x} , the two functions $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{V}(\mathbf{X}, t)$ are different. We indicate this difference by using different symbols, \mathbf{v} and \mathbf{V} .

This practice, however, is not always convenient. Often we simply use the same symbol for the velocity, and then indicate the independent variables: $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{X}, t)$. They represent the same physical object: the velocity of material particle \mathbf{X} at time t .

Let $G(\mathbf{X}, t)$ be a function of material particle and time. For example, G can be the temperature of material particle \mathbf{X} at time t . The rate of change in temperature of the material particle is

$$\frac{\partial G(\mathbf{X}, t)}{\partial t}.$$

This rate is known as the *material time derivative*.

We can calculate the material time derivative by an alternative approach. Change the variable from \mathbf{X} to \mathbf{x} by using the function $\mathbf{X}(\mathbf{x}, t)$, and write

$$g(\mathbf{x}, t) = G(\mathbf{X}, t)$$

Using chain rule, we obtain that

$$\frac{\partial G(\mathbf{X}, t)}{\partial t} = \frac{\partial g(\mathbf{x}, t)}{\partial t} + \frac{\partial g(\mathbf{x}, t)}{\partial x_i} \frac{\partial x_i(\mathbf{X}, t)}{\partial t}.$$

Thus, we can calculate the material time derivative from

$$\frac{\partial G(\mathbf{X}, t)}{\partial t} = \frac{\partial g(\mathbf{x}, t)}{\partial t} + \frac{\partial g(\mathbf{x}, t)}{\partial x_i} v_i(\mathbf{x}, t).$$

A shorthand notation for the time derivative of a function of material particles is

$$\frac{Dg}{Dt} = \frac{\partial G(\mathbf{X}, t)}{\partial t} = \frac{\partial g(\mathbf{x}, t)}{\partial t} + \frac{\partial g(\mathbf{x}, t)}{\partial x_i} v_i(\mathbf{x}, t).$$

In particular, the acceleration of a material particle is

$$a_i(\mathbf{x}, t) = \frac{\partial v_i(\mathbf{x}, t)}{\partial t} + \frac{\partial v_i(\mathbf{x}, t)}{\partial x_j} v_j(\mathbf{x}, t).$$

Compatibility of geometry. The rate of deformation relates to the gradient of velocity as

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

This relation expresses the compatibility of geometry.

Balance of forces. Consider a small volume in the body. The balance of forces now also include the inertial force:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{Dv_i}{Dt}.$$

Conservation of mass. Consider a fixed volume in space. The conservation of mass requires that the increase in the mass in the volume equals the mass flowing into the volume:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_k)}{\partial x_k} = 0.$$

Conservation of energy. Consider a fixed volume in space. The conservation of energy requires that the increase in the energy in the volume equals work done by the stress, the energy transferred from the reservoir, and the energy transferred from neighboring parts of the fluid through convection and conduction:

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u v_k)}{\partial x_k} = \sigma_{ij} D_{ij} + Q - \frac{\partial q_i}{\partial x_i}.$$

Using the conservation of mass, we simplify the conservation of energy as

$$\rho \frac{Du}{Dt} = \sigma_{ij} D_{ij} + Q - \frac{\partial q_i}{\partial x_i}.$$

Production of entropy. Thermodynamics requires that the entropy of an isolated system should never decrease. Some people wish to say that the isolated system produces entropy. Consider a fixed volume in space. List the change in the entropy in the volume, in the reservoir, and due to convection and conduction. The thermodynamic inequality takes the form:

$$\rho \frac{Ds}{Dt} - \frac{Q}{T_R} + \frac{\partial}{\partial x_k} \left(\frac{q_k}{T} \right) \geq 0.$$

Thermodynamic condition in terms of independent changes.

We now rewrite this inequality in terms of independent changes. Note that $Ds/Dt = (Du/dt + pDv/Dt)/T$ and $Dv/Dt = vD_{kk}$. Rewrite the thermodynamic inequality as

$$\frac{(\sigma_{ij} + p\delta_{ij})D_{ij}}{T} + \left(\frac{1}{T} - \frac{1}{T_R} \right) Q + q_k \frac{\partial}{\partial x_k} \left(\frac{1}{T} \right) \geq 0.$$

The inequality holds when the composite system is not in thermodynamic equilibrium, whereas the equality holds when the composite system is in thermodynamic equilibrium. The thermodynamic condition holds for three

independent and arbitrary fields: the velocity \mathbf{v} , the rate of heat Q , and the heat flux \mathbf{q} . Consequently, each of the three terms must be nonnegative.

We satisfy the inequality as follow. The first term is nonnegative if we adopt the model of viscosity, with nonnegative shear and dilation viscosities. The second term is nonnegative if we require that the energy transfer in the direction from high temperature to low temperature. The last term is nonnegative if we adopt Fourier's model of thermal conduction, with nonnegative thermal conductivity.

References

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