

Perfect Plasticity

Pre-requisite. Elements of linear elasticity (<http://imechanica.org/node/205>).

Be wise, one-dimensionalize. In 1980s, at the height of the nonlinear dynamics commotion, George Carrier advised us, "be wise, linearize". Now in teaching plasticity, we cannot linearize. We'd say, be wise, one-dimensionalize. Let us focus on a material under uniaxial stress. The uniaxial stress, however, causes the material to deform in three dimensions. When we pull a rod in the longitudinal direction, the rod contracts in the transverse directions. That is, the uniaxial stress causes triaxial strains. Nothing is really one-dimensional, but the assumption of uniaxial stress simplifies the matter significantly.

We will describe an idealized model of a material under uniaxial stress. We will allow elastic, plastic, and thermal strains. We will also invoke two empirical facts: hydrostatic stress does not affect plasticity, and plastic deformation does not change volume.

We will use this model to analyze many practical situations that clearly show salient features of plasticity.

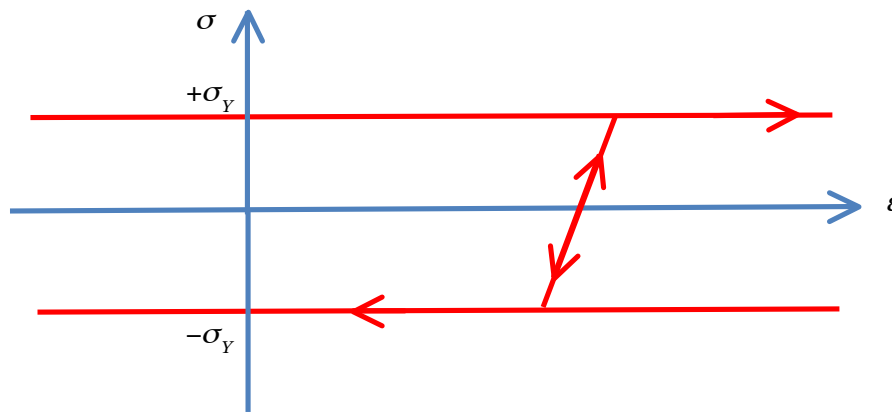
Viscous flow of a liquid vs. plastic deformation of a metal. In a liquid, molecules touch one another but frequently change neighbors. External forces cause the liquid to flow. The strain of the liquid can be indefinitely large. When a metal undergoes plastic deformation, atoms also change neighbors. In this regard, the metal behaves like a liquid. The plastic strain of the metal can also be indefinitely large.

A liquid often obeys Newton's law of viscosity: the stress is linear in the rate of deformation. The metal differs from such a viscous liquid in two important ways: the metal has yield strength, and the behavior of the metal is insensitive to the rate of deformation.

Perfect plasticity. Plasticity is a complex subject. Instead of delve into the complexity at the beginning of the course, we describe an idealized model: perfect plasticity. The word "perfect" means that the material in this model does not strain-harden—that is, the yield strength σ_y is a material property, independent of the amount of plastic strain. This salient feature of the model shows up in the stress-strain diagram of a metal subject to uniaxial stress: the diagram remains unchanged with respect to the origin of strain. The model of

perfect plasticity means that

- When the stress is within the yield strength, $-\sigma_Y < \sigma < +\sigma_Y$, the plastic strain does not change, $d\varepsilon^p = 0$.
- When the stress reaches the yield strength in tension, $\sigma = +\sigma_Y$, the plastic strain can only increase, $d\varepsilon^p > 0$. The model of perfect plasticity does not specify the amount of plastic strain. In the stress-strain diagram, this part corresponds to a horizontal line directing to the right.
- When the stress reaches the yield strength in compression, $\sigma = -\sigma_Y$, the plastic strain can only reduce, $d\varepsilon^p < 0$. In the stress-strain diagram, this part corresponds to a horizontal line directing to the left.



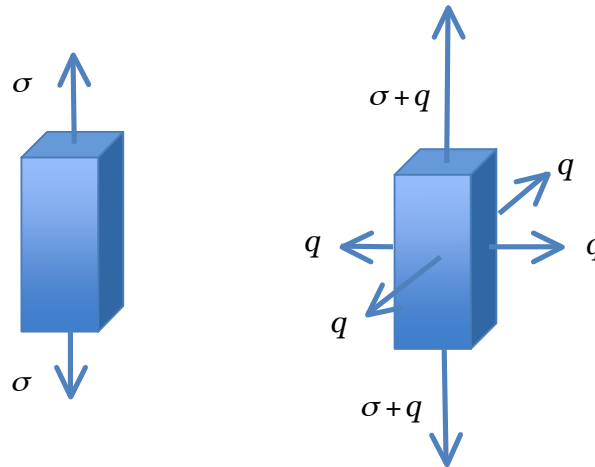
Plastic deformation does not change volume. For a metal under uniaxial stress, the two transverse plastic strains are equal, related to the longitudinal plastic strain as

$$\varepsilon_x^p = \varepsilon_y^p = -\frac{1}{2}\varepsilon_z^p.$$

Hydrostatic stress does not affect plasticity. We can superimpose a hydrostatic stress q on a uniaxial stress σ , so that the metal is under triaxial stresses:

$$\sigma_x = q, \quad \sigma_y = q, \quad \sigma_z = q + \sigma.$$

The yield condition is $|\sigma_z - \sigma_x| = \sigma_Y$, with σ_Y being independent of the superimposed hydrostatic stress q .



Rigid, perfectly plastic model. In applications like metal forming, the plastic strain is so large that elastic strain is negligible. Thus, we may neglect elastic strain, and identify the net strain entirely with plastic strain. When the stress is within the yield strength, $-\sigma_Y < \sigma < +\sigma_Y$, the material is rigid, and the strain does not change. This statement corresponds to a vertical segment in the stress-strain diagram. This model is called rigid, perfectly plastic model. We will return to this model later in this course.

Elastic, perfectly plastic model. For now we wish to include elasticity. Even though a metal is capable of arbitrarily large deformation, in many situations the plastic strain is small, on the order of elastic strain. For example, the plastic deformation of the metal can be constrained by elastic surroundings. When plastic strain and elastic strain are comparable, we need to include both in the model. This model is called elastic, perfectly plastic model.

The net strain is the sum of elastic strain and plastic strain:

$$\varepsilon = \varepsilon^e + \varepsilon^p.$$

The model has two material properties: Young's modulus E is the slope in the stress-strain diagram, and yield strength σ_Y define the range of the stress. We summarize the stress-strain diagram in three parts.

When the stress is in within the elastic range, $-\sigma_Y < \sigma < +\sigma_Y$, the stress can both increase and decrease, the increment of the elastic strain is proportional to the increment of the stress, and the plastic strain does not change:

$$d\varepsilon^e = \frac{d\sigma}{E}, \quad d\varepsilon^p = 0.$$

In the stress-strain diagram, this part corresponds to bidirectional segments of slope E .

When the stress reaches the yield strength in tension, $\sigma = +\sigma_Y$, the stress is constant, the elastic strain does not change, and the plastic strain can only increase:

$$d\varepsilon^e = 0, \quad d\varepsilon^p > 0.$$

The model of non-hardening plasticity does not specify the amount of plastic strain. In the stress-strain diagram, this part corresponds to a horizontal line directing to the right.

When the stress reaches the yield strength in compression, $\sigma = -\sigma_Y$, the stress is constant, the elastic strain does not change, and the plastic strain can only reduce:

$$d\varepsilon^e = 0, \quad d\varepsilon^p < 0.$$

The model of non-hardening plasticity does not specify the amount of plastic strain. In the stress-strain diagram, this part corresponds to a horizontal line directing to the left.

Uniaxial stress causes triaxial strains. Let the stress be along the z -axis. In the generalized Hooke's model, the two elastic strains in the transverse directions are linear in the elastic strain in the axial direction:

$$\varepsilon_x^e = \varepsilon_y^e = -\nu \varepsilon_z^e,$$

where ν is Poisson's ratio. A representative value of Poisson's ratio for metals is $\nu = 0.3$. Elastic deformation changes volume:

$$\varepsilon_x^e + \varepsilon_y^e + \varepsilon_z^e = (1 - 2\nu) \varepsilon_z^e.$$

By contrast, plastic deformation does not change volume. For a metal under uniaxial stress, the two transverse plastic strains are equal, related to the longitudinal plastic strain as

$$\varepsilon_x^p = \varepsilon_y^p = -\frac{1}{2} \varepsilon_z^p.$$

Thermal strain. We now consider the effect of temperature. We make yet another idealization: the yield strength is independent of temperature. The net strain is the sum of elastic strain, plastic strain, and thermal strain:

$$\varepsilon = \varepsilon^e + \varepsilon^p + \varepsilon^T.$$

The increment in the thermal strain is proportional to the increment in temperature:

$$d\varepsilon^T = \alpha dT,$$

where α is the coefficient of thermal expansion. The thermal strain is equal-triaxial. Now this idealized model has a total of three material properties: E , σ_y and α .

A thin metal film bonded on a thick substrate, subject to cyclic temperature. (W.D. Nix, Mechanical properties of thin films. Metallurgical Transactions A 20, 2217-2245, 1989).

- stress-temperature diagram.
- wafer curvature measurement.
- calculation
- Because hydrostatic stress does not affect plastic deformation, plastic deformation caused by uniaxial tensile stress is the same as that caused by equal-biaxial compressive stress.
- Because the substrate is much thicker than the film, the stress in the substrate is negligible.
- Strain in the film: thermal, elastic, plastic
- Strain in the substrate: thermal
- compatibility of deformation: the in-plane strain in the film equals that in the substrate.
- stress in the film is limited by the yield strength
- Constrained plastic deformation. The substrate constrains the plastic deformation of the film, so that the plastic strain is small, on the same order as the elastic and thermal strains.
- calculate the stress-temperature slope
- shakedown

Lithium-ion batteries. Kejie Zhao, Matt Pharr, Joost J. Vlassak and Zhigang Suo. [Inelastic hosts as electrodes for high-capacity lithium-ion batteries.](#) Journal of Applied Physics 109, 016110 (2011).

Lithiation and delithiation of a host. Experimental observations: a thin film does not fracture, but a thick film fractures. How can lithiation and delithiation generate a tensile stress in the film? Why does the thickness of the film matter?

A spherical metal particle embedded in a large ceramic matrix. Cooling from the processing temperature

- The metal particle is in a state of hydrostatic stress.
- The ceramic matrix is in an inhomogeneous state.
- The strain in the metal is due to elasticity and thermal expansion. No plasticity.
- The field in the ceramic matrix is given by the Lamé solution.
- Balance of force: the radial stress in the matrix, at the interface, is the same as the stress in the particle

- Incidentally, according to the Lamé solution, if the radial stress is tensile, the two hoop stresses in the matrix, at the interface, are compressive and have the magnitude half that of the radial stress. Consequently, in the matrix, at the interface, the mean stress is zero.
- Compatibility of deformation.
- The magnitude of the stress in the metal can be much larger than the yield strength of the metal.

A metallic wire in an oxide block. For a metallic wire embedded in an oxide block, a change in temperature can cause triaxial stresses, but not a hydrostatic state. The magnitude of the stress can be much larger than the yield strength. The high stress and induce voids. This phenomenon is significant in the microelectronic industry, where metallic wires are routinely embedded in dielectrics.

Consider a model problem: a thin metal wire, radius a , is embedded in an infinite block of an oxide. The composite is made at an elevated temperature T_0 , and we assume that, at this temperature, the metal wire and the oxide are stress-free. The metal has a larger coefficient of thermal expansion. When the temperature drops, the mismatch in the coefficients of thermal expansion causes the metal to develop a state of triaxial tension. We wish to determine this state of stress in the metal. To simplify the algebra, we assume that the two materials have the same Young's modulus E and the same Poisson's ratio ν .

The metal is in a homogeneous state of triaxial stresses:

$$\sigma_r = \sigma_\theta = q, \quad \sigma_z = s.$$

The metal is assumed to be elastic, perfectly plastic.

The oxide is elastic. The stress in the oxide is inhomogeneous, and is solved in the class of elasticity. This distribution is called the Lamé solution:

$$\sigma_r = A + \frac{B}{r^2}, \quad \sigma_\theta = A - \frac{B}{r^2}, \quad \sigma_z = C$$

Because the oxide is infinite, by force balance, $C = 0$. By force balance, at the surface of the oxide, the traction is $\sigma_r(a) = q$. Remote from the metal wire, the traction vanishes, $\sigma_r(\infty) = 0$. These two boundary conditions determine the two constants A and B . Thus, the stress field in the oxide is

$$\sigma_r = q \frac{a^2}{r^2}, \quad \sigma_\theta = -q \frac{a^2}{r^2}, \quad \sigma_z = 0$$

Geometric compatibility requires that at the interface $r = a$, the hoop strain and axial strain in the two materials match:

$$\begin{aligned} \varepsilon_\theta^m &= \varepsilon_\theta^{ox} \\ \varepsilon_z^m &= \varepsilon_z^{ox} \end{aligned}$$

Note that geometric compatibility does not require us to match the radial strain between the metal and the oxide.

We first assume that the metal is elastic. Let T be the current temperature. Recall the generalized Hooke's law:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) + \alpha(T - T_0).$$

Apply Hooke's law to the metal, and we obtain that

$$\varepsilon_\theta^m = \frac{1}{E}(q - \nu q - \nu s) + \alpha^m(T - T_0)$$

$$\varepsilon_z^m = \frac{1}{E}(s - 2\nu q) + \alpha^m(T - T_0)$$

$$\varepsilon_\theta^{\alpha\alpha} = \frac{1}{E}(-q - \nu q) + \alpha^{\alpha\alpha}(T - T_0)$$

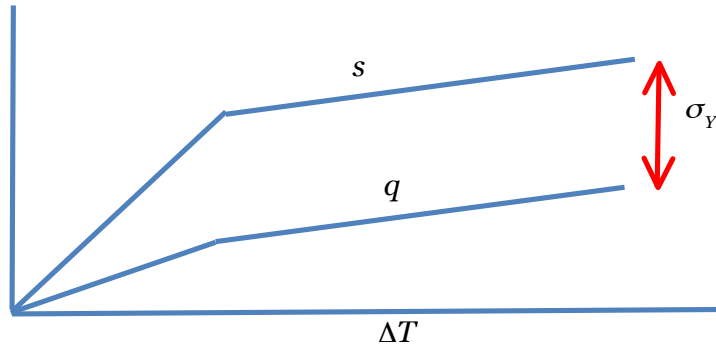
$$\varepsilon_z^{\alpha\alpha} = \alpha^{\alpha\alpha}(T - T_0)$$

Matching the strains, we obtain that

$$s = \frac{E\Delta\alpha\Delta T}{1 - \nu}, \quad q = \frac{s}{2}.$$

The metal wire yields when $s - q = \sigma_Y$, namely

$$\frac{E\Delta\alpha\Delta T}{1 - \nu} = 2\sigma_Y.$$



Now we assume that metal undergoes plastic deformation. The yield condition is

$$s - q = \sigma_Y,$$

or

$$ds - dq = 0$$

The net strains in the metal now include plastic strains:

$$d\varepsilon_\theta^m = \frac{1}{E}(dq - \nu dq - \nu ds) + \alpha^m dT + d\varepsilon_\theta^p$$

$$d\varepsilon_z^m = \frac{1}{E}(ds - 2\nu dq) + \alpha^m dT + d\varepsilon_z^p$$

Plastic deformation does not change volume, so that $\varepsilon_\theta^p = -\varepsilon_z^p / 2$. Geometric compatibility requires that $\varepsilon_\theta^m = \varepsilon_\theta^{\text{ox}}$ and $\varepsilon_z^m = \varepsilon_z^{\text{ox}}$, giving that

$$dq = \frac{3E\Delta\alpha}{5-4\nu} dT.$$

This example illustrates that, even with a model of perfect plasticity, the stress can be much larger than the yield strength.