

RIGID-PLASTIC FLOW

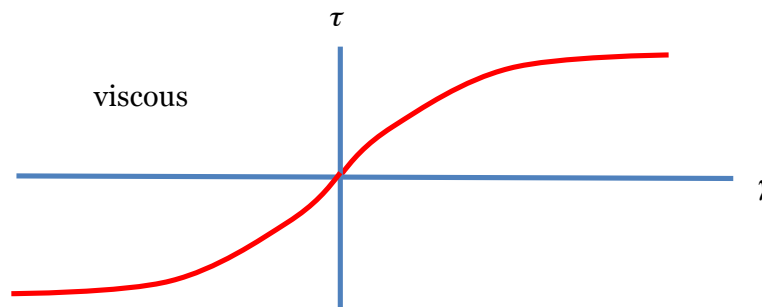
Toothpaste retains its shape when stress is small, but flows when stress is sufficiently large. Thus, the toothpaste does not drip or spread under its own weight, but extrudes from a tube or brushes onto the teeth under a gentle force. The stress above which the toothpaste flows is called the *yield stress*. Such a fluid is known as a *yield-stress fluid*, a *plastic fluid*, or a *viscoplastic fluid* (Balmforth, Frigaard, Ovarlex 2014; Coussot 2014).

Here we study plasticity in its purest form, using the *rigid-plastic model*. The model is a limit of the viscoplastic model, and is also a limit of the elastic-plastic, strain-hardening model. The rigid-plastic model characterizes a material using a single material constant, the yield stress. The model avoids complications associated with the flow, such as viscosity, elasticity, strain hardening, and thixotropy.

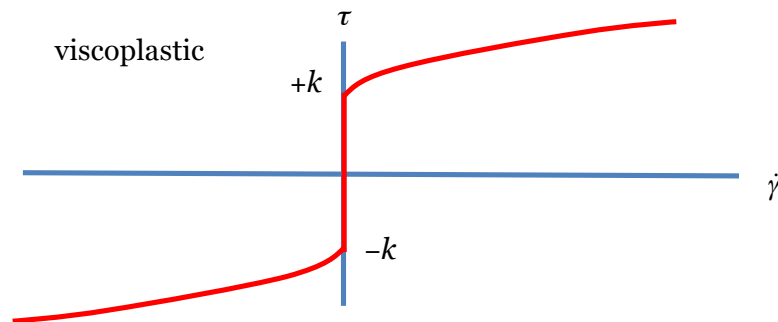
Like linear viscosity and linear elasticity, the rigid-plastic model is among the most useful rheological models. The model describes toothpastes, greases, thermoplastic polymers, and metals (Hill 1950; Tanner 2000). In particular, the rigid-plastic model describes industrial processes of metals, such as extrusion, embossing and stamping. During these processes, plastic strain is much larger than elastic strain, so that elasticity is often negligible. For pre-strained metals, hardening is also often negligible. Rigid-plastic flows have been analyzed for many geometries and loads (Druyanov and Nepershin 1994; Chakrabarty 2006).

Flow Curves

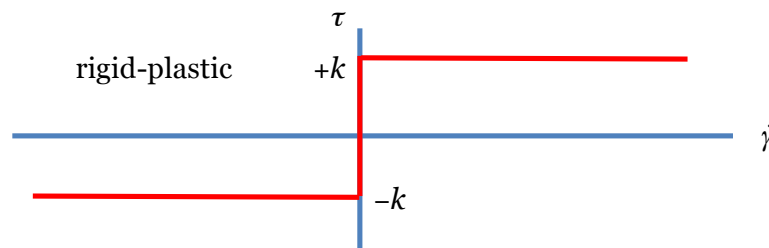
Viscous flow. We test a material under a simple state of stress, such as shear. We plot the experimental data as the *flow curve*, the relation between the shearing stress, τ , and the rate of shear, $\dot{\gamma}$. A model of viscosity specifies a one-to-one relation between the shearing stress and the rate of shear, corresponding to a monotonically increasing flow curve. When the stress vanishes, the flow stops instantly, with no delay. When the stress changes, the rate of shear changes instantly, with no delay.



Viscoplastic flow. In a viscoplastic flow, the relation between the shearing stress and the rate of shear is not one-to-one. Let k be the yield stress. When the shearing stress is within the yield stress, $|\tau| < k$, the fluid is rigid, $\dot{\gamma} = 0$. When the shearing stress exceeds the yield stress, $|\tau| > k$, the fluid flows. The fluid may flow in a complicated way. It may shear-thin, strain-harden, or be thixotropic.



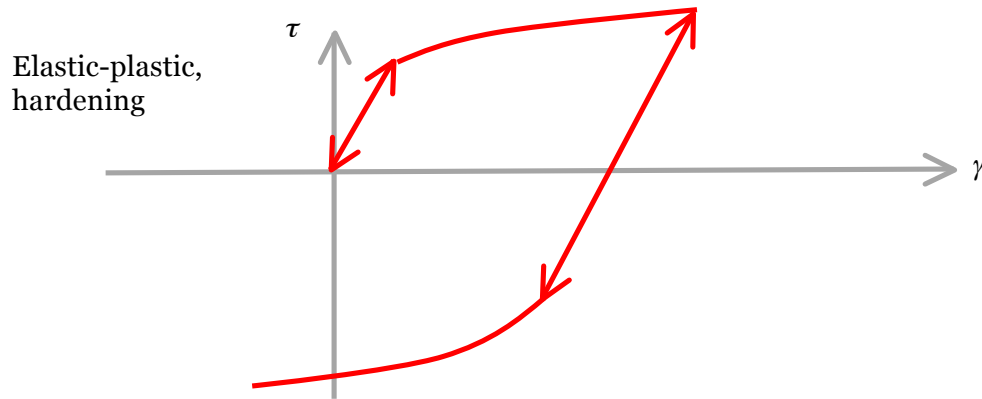
Rigid-plastic flow. If we are not interested in the complex behavior of the flow, we can use a particularly simple rheological model, the *rigid-plastic model*. The model has a single material constant: the yield stress k . In the rigid-plastic flow, the relation between the stress and the rate of deformation is not one-to-one. The model forbids the shearing stress to exceed the yield stress. When the shearing stress is within the yield stress, $|\tau| < k$, the material is rigid, $\dot{\gamma} = 0$. When the shearing stress equals the yield stress, $|\tau| = k$, the material flows, but the rate of deformation can be of any value, so long as the material flows in the direction of the stress.



In the limit of vanishing viscosity during flow, the viscoplastic model reduces to the rigid-plastic model. In the limit of vanishing yield stress, the viscoplastic model reduces to the viscous model.

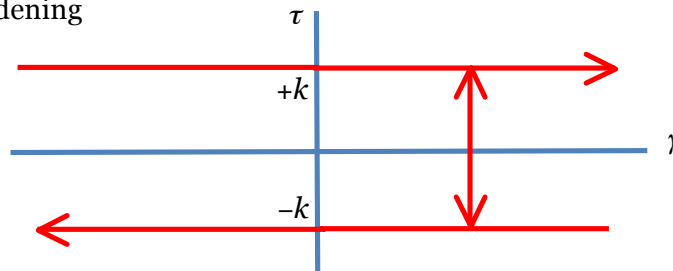
Time-independent plasticity. We have studied an elastic-plastic model, which prescribes a time-independent relation between stress and strain.

In the limit of negligible elastic strain and strain hardening, the elastic-plastic model reduces to the rigid-plastic model. The rigid-plastic fluid is also known as the *rigid, perfectly plastic material*.



We can represent the rigid-plastic model using the stress τ and strain γ . When the stress is within the yield stress, $|\tau| < k$, the material is rigid; a change in stress does not change strain. When the stress is at the yield stress, $|\tau| = k$, the model specifies the direction of strain, but not the magnitude of strain.

rigid-plastic,
non-hardening



Rigid-plastic flow in a tube. We have analyzed power-law flow in a tube (Suo 2014). We now consider rigid-plastic flow. A fluid flows in a pipe of a circular cross-section, radius a and length l , subject to pressure p_H and p_L at the two ends of the pipe. Picture a part of the fluid, a cylinder of radius r and length l , in a free-body diagram. On the surface of the cylinder, the shearing stress has a constant magnitude, τ , pointing in the direction that balances the difference in pressure at the two ends of the cylinder:

$$2\pi r l \tau = \pi r^2 (p_H - p_L).$$

Rearranging, we obtain that

$$\tau = \frac{r(p_H - p_L)}{2l}.$$

Thus, the balance of forces determines the distribution of stress in the fluid. The boundary-value problem is statically determinate.

The shearing stress is linear in r , and is the largest at the wall of the tube:

$$\tau(a) = \frac{a(p_H - p_L)}{2l}.$$

When $\tau(a) < k$, the fluid does not flow. When $\tau(a) = k$, the fluid flows. The corresponding difference in pressure is

$$p_H - p_L = \frac{2l}{a}k.$$

This expression gives the *limit load*. When the applied difference in pressure is below the limit load, the rate of flow vanishes. When the applied difference in pressure is at the limit load, the rate of flow can be of any value. The applied difference in pressure cannot exceed the limit load. Even for a fluid of a small yield stress, a tube of a large ratio l/a will require a large difference in pressure to move the fluid. The flow is localized at the wall. The interior of the fluid remains rigid, moving like a plug. The rigid-plastic model predicts the same behavior as the power-law model in the highly shear-thinning limit.

Rigid-plastic flow of a spherical pressure vessel. A spherical vessel, inner radius a and outer radius b , is subject to an inner pressure p (relative to the ambient pressure). We assume a flow of spherical symmetry. Each material particle is subject to triaxial stresses: the radial stress, $\sigma_r(r)$, and two components of the circumferential stress, $\sigma_\theta(r)$. Superposing a state of hydrostatic stress does not affect the flow. Superposing a state of hydrostatic stress $(-\sigma_\theta, -\sigma_\theta, -\sigma_\theta)$ on the original state $(\sigma_r, \sigma_\theta, \sigma_\theta)$, we obtain a state of uniaxial stress, $(\sigma_r - \sigma_\theta, 0, 0)$. The internal pressure causes the vessel to expand, so that $\sigma_r - \sigma_\theta < 0$. Thus, each material particle flows under a uniaxial compressive stress,

$$\sigma_r - \sigma_\theta = -Y,$$

where Y is the yield stress in uniaxial compression.

The balance of forces requires that

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0.$$

Combining the yield condition and the balance of forces, we obtain that

$$\frac{d\sigma_r}{dr} = \frac{2Y}{r}.$$

Integrating over the interval $a < r < b$, we obtain that

$$p = 2Y \log(b/a).$$

This is the pressure needed to cause the vessel to flow. When the applied pressure is below the limiting value, $p < 2Y \log(b/a)$, the vessel is rigid. When the pressure reaches the limiting value, $p = 2Y \log(b/a)$, the vessel flows. We can also obtain the distribution of the stresses:

$$\begin{aligned}\sigma_r(r) &= -2Y \log(b/r), \\ \sigma_\theta(r) &= Y - 2Y \log(b/r).\end{aligned}$$

The boundary-value problem is statically determinate.

The field is taken to be of spherical symmetry, so that the velocity is in the radial direction, and is a function of the radial distance, $v(r)$. The material is incompressible, so that

$$v(r) = v(a) \frac{a^2}{r^2}.$$

The problem is also kinematically determinate. Once the velocity of the inner surface, $v(a)$ is known, the velocity everywhere is known.

The problem is a moving-boundary problem. The radii of the inner and outer surfaces change with time, $a(t)$ and $b(t)$. They relate to the velocity field as

$$\frac{da(t)}{dt} = v(a), \quad \frac{db(t)}{dt} = v(b).$$

Given a history of the inner radius, $a(t)$, the history of the outer radius $b(t)$ is determined by the incompressibility:

$$[b(t)]^3 - [a(t)]^3 = [b(0)]^3 - [a(0)]^3.$$

The history of pressure that sustains the prescribed growth is

$$p(t) = 2Y \log \left[\frac{b(t)}{a(t)} \right].$$

Eliminating $b(t)$ from the last two expressions, we obtain that

$$p = \frac{2}{3} Y \log \left\{ 1 + \frac{[b(0)]^3 - [a(0)]^3}{a^3} \right\}.$$

The limiting pressure decreases as the vessel expands.

Flow under General State of Stress

Viscous flow. We have studied the viscous flow under a general state of stress (Suo 2014). Here we list the key results. Let σ_{ij} be a state of stress. The mean stress is $\sigma_m = \sigma_{kk} / 3$, and the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$. A model of viscosity specifies a one-to-one relation between the state of deviatoric stress, \mathbf{s} , and the rate of deformation, \mathbf{D} :

$$\mathbf{s} = f(\mathbf{D}).$$

In the second-invariant model of viscosity, this relation takes the form

$$\mathbf{D} = \frac{\mathbf{s}}{2\eta}.$$

For a linearly viscous fluid, the viscosity η is a material constant. For a nonlinearly viscous fluid, the viscosity η is a function of the second invariant $s_{ij}s_{ij} / 2$. The use of the invariant, as well as the form of the equation, $\mathbf{D} = \mathbf{s} / 2\eta$, ensures that the model describes an *isotropic material*. The deviatoric stress is traceless, $s_{kk} = 0$, so that the model describes an *incompressible material*, $D_{kk} = 0$.

Define the equivalent shearing stress by

$$\tau_e = \sqrt{s_{ij}s_{ij} / 2}.$$

The equivalent shearing stress is just another way to write the second invariant of the deviatoric stress. Thus, the second-invariant model assumes that the viscosity is a function of the equivalent shearing stress, $\eta(\tau_e)$.

We fix the function $\eta(\tau_e)$ by a flow curve determined experimentally under one type of stress, such as shearing stress. Let τ be the shearing stress, $\dot{\gamma}$ be the rate of shear, and $\dot{\gamma} = h(\tau)$ be the flow curve measured under the shearing stress. When the fluid is under any state of stress, the second-invariant model predicts that the rate of deformation is

$$\mathbf{D} = \frac{h(\tau_e)}{2\tau_e} \mathbf{s}.$$

So long as the flow curve $\dot{\gamma} = h(\tau)$ is a monotonically increasing function, the model $\mathbf{D} = \mathbf{s} / 2\eta$ specifies a one-to-one relation between the rate of deformation and the deviatoric stress.

Viscoplastic flow. The model of viscosity is applicable even when the flow curve $\dot{\gamma} = h(\tau)$ is not a one-to-one relation between the shearing stress and the rate of shear. A viscoplastic fluid has a particular type of flow curve, with a yield stress k . For example, a function often used to fit the experimentally measured flow curve is

$$h(\tau) = \begin{cases} 0, & \text{for } 0 < \tau < k \\ \left(\frac{\tau - k}{A} \right)^n, & \text{for } \tau > k \end{cases}$$

The flow curve is not a one-to-one relation: A range of stress $0 < \tau < k$ gives a single value of the rate of shear, $\dot{\gamma} = 0$.

For a fluid under a general state of stress, define the equivalent shearing stress:

$$\tau_e = \sqrt{s_{ij}s_{ij}/2}.$$

The second-invariant model assumes the Mises yield condition:

$$\frac{1}{2}s_{ij}s_{ij} = k^2.$$

The model predicts that the rate of deformation relates to the stress as

$$\mathbf{D} = \frac{h(\tau_e)}{2\tau_e} \mathbf{s}.$$

Rigid-plastic flow. The rigid-plastic flow is a limit of the viscoplastic flow. The second-invariant model assumes the *Mises yield condition*:

$$\frac{1}{2}s_{ij}s_{ij} = k^2,$$

where k is the yield stress of the fluid in shear. When $s_{ij}s_{ij}/2 < k^2$, the material is rigid, $\mathbf{D} = \mathbf{0}$. When $s_{ij}s_{ij}/2 = k^2$, the material flows according to the *Levy-Mises flow rule*:

$$\mathbf{D} = \lambda \mathbf{s}.$$

The pre-factor λ is a positive scalar.

The Levy-Mises flow rule, $\mathbf{D} = \lambda \mathbf{s}$, looks similar to the model of viscosity, $\mathbf{D} = \mathbf{s}/2\eta$, but the difference is important. The Levy-Mises flow rule does not specify the magnitude of the rate of deformation, λ . By contrast, the model of viscosity specifies η as a function of the second invariant of the deviatoric stress, so that the model of viscosity specifies a one-to-one relation between the deviatoric stress and the rate of deformation.

The use of the invariant ensures that the model describes an isotropic material. Because the deviatoric stress is traceless, $s_{kk} = 0$, the Levy-Mises flow rule ensures incompatibility, $D_{kk} = 0$. The model satisfies the thermodynamic inequality $s_{ij}D_{ij} \geq 0$ for any state of stress.

Rigid-plastic flow under a given state of stress. For a given state of stress, the rigid-plastic model does not specify the magnitude of the rate of deformation, but specifies the direction of the rate of deformation. The Levy-Mises flow rule, $\mathbf{D} = \lambda \mathbf{s}$ and $\lambda > 0$, is equivalent to the following relations

$$\frac{D_{11}}{s_{11}} = \frac{D_{22}}{s_{22}} = \frac{D_{33}}{s_{33}} = \frac{D_{23}}{s_{23}} = \frac{D_{31}}{s_{31}} = \frac{D_{12}}{s_{12}} > 0.$$

Rigid-plastic flow with prescribed rate of deformation.

Conversely, given a rate of deformation \mathbf{D} , with $D_{kk} = 0$, the rigid-plastic model determines the state of deviatoric stress. So long as $\mathbf{D} \neq \mathbf{0}$, the material flows. The state of stress simultaneously satisfies the Mises yield condition, $s_{ij}s_{ij}/2 = k^2$, and the Levy-Mises flow rule, $\mathbf{D} = \lambda \mathbf{s}$. The flow rule implies that $D_{ij}D_{ij} = \lambda^2 s_{ij}s_{ij}$. This expression, together with the yield condition, gives that $\lambda = \sqrt{D_{ij}D_{ij}} / \sqrt{2}k$. The flow rule becomes that

$$\mathbf{s} = k \sqrt{\frac{2}{D_{ij}D_{ij}}} \mathbf{D}.$$

When the rate of deformation changes with time, $\mathbf{D}(t)$, the state of deviatoric stress changes instantly, with no delay, moving on the yield surface.

But the state of deviatoric stress will remain fixed for a special history of deformation. When all components of the rate of deformation change proportionally, we write

$$\mathbf{D}(t) = \alpha(t) \mathbf{D}^*,$$

where \mathbf{D}^* is a fixed rate of deformation, and $\alpha(t)$ is a scalar function of time. The rigid-plastic model predicts the deviatoric stress

$$\mathbf{s} = k \sqrt{\frac{2}{D_{ij}^*D_{ij}^*}} \mathbf{D}^*$$

The state of deviatoric stress remains fixed when all components of deformation change proportionally. This result generalizes the shearing flow and elongational flow.

Concurrent shear and elongation. Consider a thin-walled tube, of a rigid-plastic material, subject to both axial force and torque. We prescribe the histories of shearing strain and tensile strain—that is, we prescribe them as functions of time, $\gamma(t)$ and $\varepsilon(t)$. We now use the second-invariant model to predict the history of the shearing stress and tensile stress, $\tau(t)$ and $\sigma(t)$.

The state of stress is

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The mean stress is $\sigma_m = \sigma/3$, and the state of deviatoric stress is

$$\mathbf{s} = \begin{bmatrix} 2\sigma/3 & \tau & 0 \\ \tau & -\sigma/3 & 0 \\ 0 & 0 & -\sigma/3 \end{bmatrix}.$$

The second invariant of the deviatoric stress is

$$\frac{1}{2} s_{ij} s_{ij} = \frac{1}{3} \sigma^2 + \tau^2.$$

Let k be the yield stress under shear. The Mises yield condition is

$$\frac{1}{3} \sigma^2 + \tau^2 = k^2.$$

This yield condition corresponds to an ellipse on the plane with axes σ and τ .

The rate of deformation is

$$\mathbf{D} = \begin{bmatrix} \dot{\varepsilon} & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & -\dot{\varepsilon}/2 & 0 \\ 0 & 0 & -\dot{\varepsilon}/2 \end{bmatrix},$$

where the rate of shear is $\dot{\gamma} = d\gamma(t)/dt$ and the rate of elongation is $\dot{\varepsilon} = d\varepsilon(t)/dt$. (We assume that ε is the natural strain.)

The Levy-Mises flow rule $\mathbf{D} = \lambda \mathbf{s}$ reduces to a single independent equation:

$$\frac{\sigma}{\tau} = 3 \frac{\dot{\varepsilon}}{\dot{\gamma}}$$

When the fluid flows, the state of stress (τ, σ) stays on the yield locus.

Given the histories of the strains, $\gamma(t)$ and $\varepsilon(t)$, the yield condition and the Levy-Mises flow rule determine the history of stresses:

$$\tau^2 = \frac{k^2}{3(\dot{\epsilon}/\dot{\gamma})^2 + 1}, \quad \sigma^2 = \frac{3(\dot{\epsilon}/\dot{\gamma})k^2}{3(\dot{\epsilon}/\dot{\gamma})^2 + 1}.$$

The signs of the two stresses are determined by the signs of the rates $\dot{\gamma}$ and $\dot{\epsilon}$. For the rigid-plastic model, the state of stress at a particular time depends only on the rate of deformation at this time. When the deformation is proportional, the ratio $\dot{\epsilon}/\dot{\gamma}$ is fixed, so is the state of stress (τ, σ) .

Inhomogeneous deformation. When a body of the material undergoes inhomogeneous deformation, we regard the body as a sum of many small pieces. Each small piece undergoes homogeneous deformation, and obeys the rheological model specified above. Different pieces in the body communicate through the compatibility of deformation and the balance of forces.

Compatibility of deformation. Let \mathbf{x} be the coordinate of a place in space, t be the time, and $v_i(\mathbf{x}, t)$ be the velocity of a small piece of fluid at the place \mathbf{x} and time t . The compatibility of deformation relates the rate of deformation to the field of velocity:

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

This relation between the rate of deformation and velocity holds for deformation of any magnitude.

Balance of forces. Let $b_i(\mathbf{x}, t)$ be the body force per unit volume, and ρ be the mass per unit volume. For an incompressible fluid, ρ is constant. Inside the body, the balance of forces relates the stress to the body force and the inertial force:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right).$$

The right-hand side is the inertial force per unit volume.

Let $n_i(\mathbf{x}, t)$ be the unit vector normal to a small part of the surface of the body. Let $t_i(\mathbf{x}, t)$ be the traction, i.e., the force per unit area acting on the surface. On the surface of the body, the balance of forces relates the stress to the traction:

$$\sigma_{ij} n_j = t_i.$$

Boundary conditions. We divide the surface of the body into two parts. On one part of the surface, S_v , we prescribe the velocity of the fluid. On the other part of the surface, S_t , we prescribe the traction. These boundary conditions,

along with the equations listed above, constitute a boundary-value problem that governs the inhomogeneous deformation in the body.

Eulerian approach. Rigid-plastic flow is analyzed in the same way as viscous flow. We use the Eulerian approach. We often neglect inertia, so that the model has a single material constant: the yield stress k . The stress scales with the yield stress, and the field is independent of the material constant, and is fully governed by the geometry and loads. When the geometry and loads change, the flow changes instantly, with no delay.

Scaling of rigid-plastic flow. Consider a rigid-plastic flow of a body, characterized by a length a and load σ_{appl} . The load that causes the body to flow scales with the yield stress, $\sigma_{\text{appl}} \propto k$. The fields of stress, rate of deformation, and velocity take the form

$$\begin{aligned}\sigma_{ij} &= k \hat{\sigma}_{ij} \left(\frac{\mathbf{x}}{a} \right), \\ D_{ij} &= \beta \hat{D}_{ij} \left(\frac{\mathbf{x}}{a} \right), \\ v_i &= \beta a \hat{v}_i \left(\frac{\mathbf{x}}{a} \right).\end{aligned}$$

Here β is an arbitrary scalar, and $\hat{\sigma}_{ij}$, \hat{D}_{ij} and \hat{v}_i stand for dimensionless functions. The dimensionless functions also depend on ratios that specify the shape of the body.

Proof. The above form satisfies all the governing equations. The boundary-value problems are governed by the following equations. We neglect the effect of inertia and the body force. Under these conditions, the balance of forces takes the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0.$$

Compatibility of deformation requires that

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

The rigid-plastic flow satisfies the Mises yield condition,

$$\frac{1}{2} s_{ij} s_{ij} = k^2,$$

and the Levy-Mises flow rule,

$$\mathbf{D} = \lambda \mathbf{s}.$$

Rigid-Plastic Flow under the Plane Strain Conditions

Compatibility of geometry. When a body flows under the plane strain conditions, the in-plane velocities are functions of two coordinates of the plane, $v_1(x_1, x_2)$ and $v_2(x_1, x_2)$, and the out-of-plane velocity vanishes, $v_3 = 0$. The non-zero components of the rate of deformation relate to the field of velocity by

$$D_{11} = \frac{\partial v_1}{\partial x_1}, \quad D_{22} = \frac{\partial v_2}{\partial x_2}, \quad D_{12} = \frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right).$$

Rheology of material. Under stress σ_{ij} , the mean stress is $\sigma_m = \sigma_{kk} / 3$, and the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$. The components of the deviatoric stress are

$$s_{11} = \sigma_{11} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3},$$

$$s_{22} = \sigma_{22} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3},$$

$$s_{33} = \sigma_{33} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3},$$

$$s_{12} = \sigma_{12},$$

$$s_{13} = \sigma_{13},$$

$$s_{23} = \sigma_{23}.$$

We assume the material to be incompressible, isotropic and rigid-plastic. That is, the material obeys the Levy-Mises flow rule:

$$D_{ij} = \lambda s_{ij}.$$

When $s_{ij}s_{ij}/2 < k^2$, the material is rigid, and $\lambda = 0$. When $s_{ij}s_{ij}/2 = k^2$, the material flows, and $\lambda > 0$. Here k is the yield stress of the material in shear.

When the material flows under the plane strain conditions, $D_{13} = D_{23} = D_{33} = 0$ and that $s_{13} = s_{23} = s_{33} = 0$. The condition $s_{33} = 0$ gives that

$$\sigma_{33} = \frac{\sigma_{11} + \sigma_{22}}{2}.$$

Consequently, the mean stress becomes

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22}}{2},$$

and the nonzero components of the deviatoric stress are

$$s_{11} = \frac{\sigma_{11} - \sigma_{22}}{2},$$

$$s_{22} = \frac{\sigma_{22} - \sigma_{11}}{2},$$

$$s_{12} = \sigma_{12}.$$

Under the plane strain conditions, the Mises yield condition is

$$\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 = k^2,$$

and the Levy-Mises flow rule becomes that

$$\frac{\partial v_1}{\partial x_1} = \lambda \frac{\sigma_{11} - \sigma_{22}}{2},$$

$$\frac{\partial v_2}{\partial x_2} = \lambda \frac{\sigma_{22} - \sigma_{11}}{2},$$

$$\frac{1}{2} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) = \lambda \sigma_{12}.$$

Balance of forces. The nonzero components of stress are $\sigma_{11}(x_1, x_2)$, $\sigma_{22}(x_1, x_2)$, $\sigma_{12}(x_1, x_2)$, and $\sigma_{33}(x_1, x_2)$. The balance of forces requires that

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0,$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0.$$

We have neglected the body force and inertia. On the surface of the body, the stress relates to the traction as

$$\sigma_{11} n_1 + \sigma_{12} n_2 = t_1,$$

$$\sigma_{21} n_1 + \sigma_{22} n_2 = t_2.$$

Summary of governing equations. The rigid-plastic flow under the plane strain conditions corresponds to a boundary-value problem. The Mises yield condition gives an algebraic equation:

$$\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 = k^2.$$

The balance of forces gives two partial differential equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0,$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0.$$

Eliminating λ from the Levy-Mises flow rule, we obtain that

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0,$$

$$\frac{\frac{\partial v_1}{\partial x_1}}{\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}} = \frac{\sigma_{11} - \sigma_{22}}{4\sigma_{12}}.$$

The five equations, along with the boundary conditions, govern the five fields:

$$\sigma_{11}(x_1, x_2), \sigma_{22}(x_1, x_2), \sigma_{12}(x_1, x_2), v_1(x_1, x_2) \text{ and } v_2(x_1, x_2).$$

State of stress. Let us review essential ideas of stress. We define a state of stress, σ , as a function that maps one vector to another vector:

$$\mathbf{f} = \sigma(\mathbf{a}),$$

where \mathbf{a} is the area vector of a planar region, and \mathbf{f} is the force acting on the region. The balance of forces requires that this function be a *linear map*. In linear algebra, a linear map between two vectors is an example of *tensor*. Thus, the state of stress is a tensor that maps the area vector of a planar region to the force acting on the region. This statement links a state of stress to the algebra of tensor.

Components of a state of stress relative to a basis. A state of stress is a physical fact, independent of the choice of basis. We next choose a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We will always use orthonormal basis, so that

$$\mathbf{e}_k \cdot \mathbf{e}_j = \delta_{kj}.$$

Consider a unit cube of the material with faces normal to the base vectors. Acting on the face normal to \mathbf{e}_j is the force $\sigma(\mathbf{e}_j)$. This force is a vector, and is a linear combination of the three base vectors:

$$\sigma(\mathbf{e}_j) = \sigma_{ij} \mathbf{e}_i,$$

The nine coefficients σ_{ij} are called the components of the state of stress, σ , relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. By convention, the second index of σ_{ij} indicates the direction of the vector normal to the face, and the first index indicates the direction of the component of the force. That is, σ_{ij} is the component of the

force, in the direction \mathbf{e}_i , acting on the face of the unit cube normal to \mathbf{e}_j . The balance of moment requires that the matrix be symmetric, $\sigma_{ij} = \sigma_{ji}$.

Change of basis. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be one orthonormal basis, and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be another orthonormal basis. The vector \mathbf{e}'_a is a linear combination of the base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Write

$$\mathbf{e}'_a = Q_{aj} \mathbf{e}_j,$$

where Q_{aj} is the *matrix of transformation*. Taking the inner product of the above equation with \mathbf{e}_k , and recalling that $\mathbf{e}_k \cdot \mathbf{e}_j = \delta_{kj}$, we obtain that

$$\mathbf{e}'_a \cdot \mathbf{e}_k = Q_{ak}.$$

Thus, Q_{aj} is the cosine of the angle between the two the unit vectors, \mathbf{e}'_a and \mathbf{e}_k . We can also express the vector \mathbf{e}_j as a linear combination of $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. The component of \mathbf{e}_j in the direction \mathbf{e}'_a is Q_{aj} , namely,

$$\mathbf{e}_j = Q_{aj} \mathbf{e}'_a.$$

In $\mathbf{e}'_a = Q_{aj} \mathbf{e}_j$, replacing \mathbf{e}_j with $\mathbf{e}_j = Q_{bj} \mathbf{e}'_b$, we obtain that

$$Q_{aj} Q_{bj} = \delta_{ab}.$$

Such a matrix is known as an *orthogonal matrix*.

We will only be interested in a special case. Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be the basis that rotates by an angle ϕ from the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ around \mathbf{e}_3 . The two sets of base vectors relate as

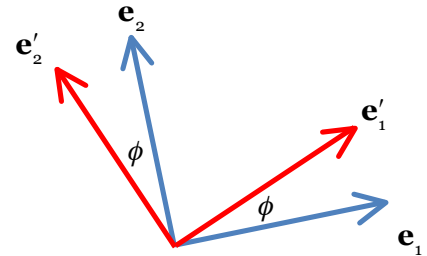
$$\mathbf{e}'_1 = \mathbf{e}_1 \cos \phi + \mathbf{e}_2 \sin \phi,$$

$$\mathbf{e}'_2 = -\mathbf{e}_1 \sin \phi + \mathbf{e}_2 \cos \phi$$

$$\mathbf{e}'_3 = \mathbf{e}_3.$$

Consequently, the matrix of transformation is

$$[Q_{aj}] = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



For a given state of stress, its components change when the basis changes. A state of stress σ is a physical fact, and is independent of the choice of the basis. Given a basis, we picture a unit cube with edges coinciding

with the directions of the three base vectors. The components of the stress are the forces acting on the faces of the unit cube. A different basis corresponds to a unit cube in a different orientation. Thus, for a given state of stress, its components change when the basis changes.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be one orthonormal basis. Write a state of stress σ as

$$\sigma(\mathbf{e}_j) = \sigma_{ij} \mathbf{e}_i.$$

The coefficients σ_{ij} are the components of the state of stress σ relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be another basis. Write the same state of stress as

$$\sigma(\mathbf{e}'_b) = \sigma'_{ab} \mathbf{e}'_a,$$

The nine coefficients σ'_{ab} are the components of the state of stress σ relative to the basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$.

The two sets of base vectors are related through a matrix of transformation:

$$\mathbf{e}'_a = Q_{aj} \mathbf{e}_j, \quad \mathbf{e}_j = Q_{aj} \mathbf{e}'_a$$

Recall that σ is a linear map. Write the following identities:

$$\sigma(\mathbf{e}'_b) = \sigma(Q_{bk} \mathbf{e}_k) = Q_{bk} \sigma(\mathbf{e}_k) = Q_{bk} \sigma_{jk} \mathbf{e}_j = Q_{bk} \sigma_{jk} Q_{aj} \mathbf{e}'_a$$

A comparison of $\sigma(\mathbf{e}'_b) = Q_{bk} \sigma_{jk} Q_{aj} \mathbf{e}'_a$ and $\sigma(\mathbf{e}'_b) = \sigma'_{ab} \mathbf{e}'_a$ gives that

$$\sigma'_{ab} = Q_{aj} Q_{bk} \sigma_{jk}.$$

This equation relates the components of one state of stress relative to two sets of base vectors. Similarly, we write the inverse relation:

$$\sigma_{jk} = Q_{aj} Q_{bk} \sigma'_{ab}.$$

In-plane components of a state of stress. Consider a coordinate (α, β) at an angle ϕ from the coordinate (x_1, x_2) . For a given state of stress, its in-plane components relative to the basis of (α, β) are

$$\begin{bmatrix} \sigma_{\alpha\alpha} & \sigma_{\alpha\beta} \\ \sigma_{\alpha\beta} & \sigma_{\beta\beta} \end{bmatrix},$$

and its in-plane components relative to the basis of (x_1, x_2) are

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}.$$

The two sets of components of the same state of stress are related as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \sigma_{\alpha\alpha} & \sigma_{\alpha\beta} \\ \sigma_{\alpha\beta} & \sigma_{\beta\beta} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}.$$

Multiplying the matrices, we obtain that

$$\begin{aligned} \sigma_{11} &= \frac{\sigma_{\alpha\alpha} + \sigma_{\beta\beta}}{2} + \frac{\sigma_{\alpha\alpha} - \sigma_{\beta\beta}}{2} \cos 2\phi - \sigma_{\alpha\beta} \sin 2\phi, \\ \sigma_{22} &= \frac{\sigma_{\alpha\alpha} + \sigma_{\beta\beta}}{2} - \frac{\sigma_{\alpha\alpha} - \sigma_{\beta\beta}}{2} \cos 2\phi + \sigma_{\alpha\beta} \sin 2\phi, \\ \sigma_{12} &= \frac{\sigma_{\alpha\alpha} - \sigma_{\beta\beta}}{2} \sin 2\phi + \sigma_{\alpha\beta} \cos 2\phi. \end{aligned}$$

The inverse relation is

$$\begin{aligned} \sigma_{\alpha\alpha} &= \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\phi + \sigma_{12} \sin 2\phi, \\ \sigma_{\beta\beta} &= \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\phi - \sigma_{12} \sin 2\phi, \\ \sigma_{\alpha\beta} &= -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\phi + \sigma_{12} \cos 2\phi. \end{aligned}$$

We have just obtained these results from the algebra of tensor. Recall that in the beginning we link the state stress to the algebra of tensor through the balance of forces. We can of course obtain the above results directly by the balance of forces.

Orientation of slip. Given a state of stress, its components relative to one basis can be easier to understand than its components relative to another basis. For example, under a uniaxial tension, a common choice of basis is to make one base vector coincide with the direction of the applied force. In discussing plastic flow, however, we often choose another basis, with a base vector at 45° from the direction of the axial force. Because the axial direction and the two transverse directions are three principal directions of the state of stress, the shearing stress reaches the maximum in the direction at 45° between two principal directions.

This second choice is particularly useful to us. Under the plane strain conditions, the yield condition is

$$\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 = k^2.$$

The yield condition is obtained using an invariant of stress. Consequently, the yield condition takes the same form in a coordinate of any orientation. Consider a new coordinate (α, β) , which is at an angle ϕ counterclockwise from the

coordinate (x_1, x_2) . In the coordinate (α, β) , the yield condition takes the same form:

$$\left(\frac{\sigma_{\alpha\alpha} - \sigma_{\beta\beta}}{2} \right)^2 + \sigma_{\alpha\beta}^2 = k^2.$$

We can choose the coordinate in an orientation such that $\sigma_{\alpha\alpha} = \sigma_{\beta\beta}$. In this orientation,

$$\begin{aligned} \sigma_{\alpha\alpha} &= \sigma_{\beta\beta} = \sigma_{33} = \sigma_m, \\ \sigma_{\alpha\beta} &= k. \end{aligned}$$

Thus, when the small piece of material flows, the material is subject to the hydrostatic stress σ_m and the shearing stress k . Of all orientation, this orientation has the maximum shearing stress, which equals the yield stress k . This orientation is known as the *orientation of slip*.

Because the shearing stresses on a small piece of material always form quadruples, the slip occurs in two orthogonal directions, α and β . We adopt the sign convention that $\sigma_{\alpha\beta} = +k$.

We have represented the same state of stress using components in two systems of coordinates, (x_1, x_2) and (α, β) . When the coordinate (α, β) coincide with the orientation of slip, $\sigma_{\alpha\alpha} = \sigma_{\beta\beta} = \sigma_m$ and $\sigma_{\alpha\beta} = k$. The components of this state of stress relative to the basis of (x_1, x_2) are

$$\begin{aligned} \sigma_{11} &= \sigma_m - k \sin 2\phi, \\ \sigma_{22} &= \sigma_m + k \sin 2\phi, \\ \sigma_{12} &= k \cos 2\phi. \end{aligned}$$

Slip lines. Next consider a body deforming inhomogeneously. We regard the body as a sum of many small pieces. Each small piece undergoes a homogeneous deformation, as described by the rigid-plastic model. As the body flows, the state of stress may vary from one material particle to another. The magnitude of the hydrostatic stress and the orientation of slip may vary from one material particle to another. That is, the two unknown variables are functions $\sigma_m(x_1, x_2)$ and $\phi(x_1, x_2)$. In the body, the orientation of slip forms two sets of curves, known as the *slip lines*. The two sets of slip lines may be curved, but are orthogonal to each other. The slip lines divide the body into many small squares. Each square is in the local orientation of slip.

Statically determinate rigid-plastic flow. The problem has three independent components of stress, governed by three equations. One algebraic

equation comes from the yield criterion, and two partial differential equations come from the balance of forces. Consequently, The problem is statically determinate provided the boundary conditions are traction-prescribed.

We have used the yield condition to express the three components of stress, $\sigma_{11}, \sigma_{22}, \sigma_{12}$, in terms of two variables the mean stress σ_m and the orientation of slip, ϕ , namely,

$$\sigma_{11} = \sigma_m - k \sin 2\phi,$$

$$\sigma_{22} = \sigma_m + k \sin 2\phi,$$

$$\sigma_{12} = k \cos 2\phi.$$

Inserting these expressions into the two equations of the balance of forces, we obtain that

$$\frac{\partial \sigma_m}{\partial x_1} - 2k \cos 2\phi \frac{\partial \phi}{\partial x_1} - 2k \sin 2\phi \frac{\partial \phi}{\partial x_2} = 0,$$

$$-2k \sin 2\phi \frac{\partial \phi}{\partial x_2} + \frac{\partial \sigma_m}{\partial x_2} + 2k \cos 2\phi \frac{\partial \phi}{\partial x_1} = 0.$$

The unknowns are functions of the two coordinates, taken to be the curved coordinates of the slip lines, $\sigma_m(s_\alpha, s_\beta)$ and $\phi(s_\alpha, s_\beta)$. Now take (x_1, x_2) to coincide with (α, β) at a local point, so that $\phi = 0$. The above equations become

$$\frac{\partial}{\partial s_\alpha} (\sigma_m - 2k\phi) = 0,$$

$$\frac{\partial}{\partial s_\beta} (\sigma_m + 2k\phi) = 0.$$

The solution to these equations are

$$\sigma_m - 2k\phi = f(s_\beta),$$

$$\sigma_m + 2k\phi = g(s_\alpha).$$

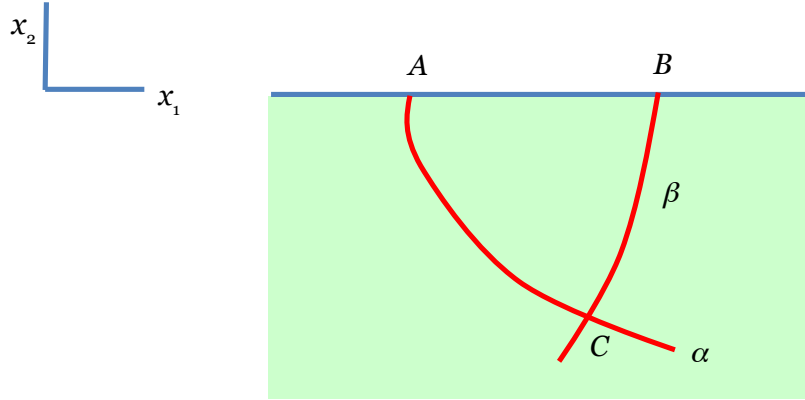
Thus, the balance of forces results in two algebraic equations.

Calculate the slip-line field. When both components of the traction are prescribed on a segment of the boundary, the stress field is determined within a triangle bounded by the boundary and two slip lines. Here are the steps.

- On a traction-prescribed boundary, two components of stress are known.
- Use the yield condition to determine the third component of stress on the boundary.
- Also on the boundary, use the transformation of tensor components to determine the direction of the maximum shear stress and the magnitude of the hydrostatic stress.

- For a point inside the body, connect the point to the boundary with two slip lines.
- Determine the orientation of the maximum shear stress and the magnitude of the hydrostatic stress by the two algebraic equations.

This procedure has been implemented for numerical calculations (Hill 1950; Chakrabarty 2006;). Here we give examples of analytical solutions.



Flat, traction-free boundary. A flat, traction-free boundary lies on the x_1 -axis, $\sigma_{12} = \sigma_{22} = 0$. If the material beneath flows, it obeys the yield condition:

$$\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2 = k^2.$$

Inserting $\sigma_{12} = \sigma_{22} = 0$, we obtain that $\sigma_{11} = \pm 2k$. That is, the material flows by uniaxial tension or compression.

Consider the case of uniaxial tension, $\sigma_{11} = +2k$. Recall the transformation of components:

$$\sigma_{11} = \sigma_m - k \sin 2\phi,$$

$$\sigma_{22} = \sigma_m + k \sin 2\phi,$$

$$\sigma_{12} = k \cos 2\phi.$$

We find that $\sigma_m = k$ and $\phi = -\pi/4$. These results are evident from the balance of forces.

So far we have determined the quantities on the boundary. Now look at a point C inside the body. Connect this point by two slip lines to two points on the boundary, A and B. Along the two slip lines, we have the algebraic equations due to the balance of forces:

$$\sigma_m(C) - 2k\phi(C) = \sigma_m(A) - 2k\phi(A)$$

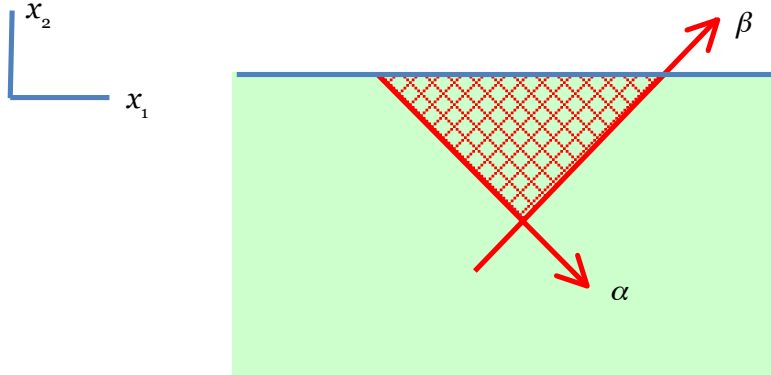
$$\sigma_m(C) - 2k\phi(C) = \sigma_m(B) - 2k\phi(B)$$

The values of the mean stress and the orientation of slip are known at the points on the boundary, so that

$$\sigma_m(C) - 2k\phi(C) = k - 2k(-\pi/4)$$

$$\sigma_m(C) - 2k\phi(C) = k - 2k(-\pi/4)$$

Solve for the two variables at point C, we obtain that $\sigma_m(C) = k$ and $\phi(C) = -\pi/4$. Thus, within a triangle beneath the flat, traction-free boundary, the state of stress is homogeneous, uniaxial tension, $\sigma_{12} = \sigma_{22} = 0$ and $\sigma_{11} = +2k$. Everywhere in the triangle, the mean stress is $\sigma_m = k$, and the orientation of slip is $\phi = -\pi/4$. The size of the triangle is set by its base, where the boundary is traction-free, and the material flows.



Compression without friction. A boundary lies on the x_1 -axis. The boundary satisfies the condition $\sigma_{12} = 0$. Thus, the boundary is normal to a principal stress, and the orientation of slip is at 45 degrees from the x_1 -axis. The field of stress and the field of velocity are both homogeneous in the triangle beneath the boundary.

$$|\sigma_{11} - \sigma_{22}| = 2k.$$

Centered fan. Assume that the α -lines are rays, and the β -lines are circular arcs. The orientation of each slip line is known, the same as the polar angle,

$$\phi = \theta.$$

All we need to do is to determine the field of the mean stress, $\sigma_m(r, \theta)$. We next invoke the two algebraic equations due to the balance of forces. Along the rays, the quantity is a function of the polar angle:

$$\sigma_m - 2k\theta = f(\theta).$$

Along the arcs, the quantity is a function of the radius:

$$\sigma_m + 2k\theta = g(r).$$

Solving the two equations, we obtain that the mean stress in the fan is inhomogeneous, given by

$$\sigma_m = c - 2k\theta,$$

where c is a constant everywhere in the fan. The constant is to be determined by matching with other boundary conditions. This stress field is singular, giving multiple values of the mean stress as the center is approached, $r \rightarrow 0$.

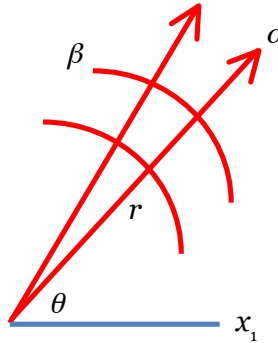
Next assume that the α -lines are circular arcs, and the β -lines are rays. The orientation of slip relates to the polar angle as

$$\phi = \theta - \frac{\pi}{2}.$$

The mean stress is

$$\sigma_m = c - 2k\theta,$$

where c is a constant everywhere in the fan.



Rigid-plastic flow under a rigid punch. We press a rigid punch into a substrate. When the applied force is small, the stress in the substrate concentrates in small zones around the edges of the punch. Within these zones, the substrate undergoes plastic flow. As the applied force increases, the plastic zones enlarge. When the plastic zones from the two edges merge, the punch can indent into the substrate with no further increase of the load. The applied force is said to have reached the *limit load*.

The slip line field consists of four triangles of homogeneous field, and two centered fans. In triangle A, the material is under uniaxial compression,

$\sigma_{12} = \sigma_{22} = 0$ and $\sigma_{11} = -2k$. The mean stress is $\sigma_m(A) = -k$, and the orientation of slip is $\phi(A) = \pi/4$.

In triangle B, the orientation of slip is $\phi(B) = -\pi/4$, but the mean stress is to be determined. Follow an α -line from a point underneath the punch all the way to a point on the traction-free boundary. Write

$$\sigma_m(B) - 2k\phi(B) = \sigma_m(A) - 2k\phi(A).$$

This equation gives the mean stress under the punch:

$$\sigma_m(B) = -(1 + \pi)k.$$

Using $\phi(B) = -\pi/4$ and $\sigma_m(B) = -(1 + \pi)k$, we can calculate the stress field in the substrate underneath the substrate:

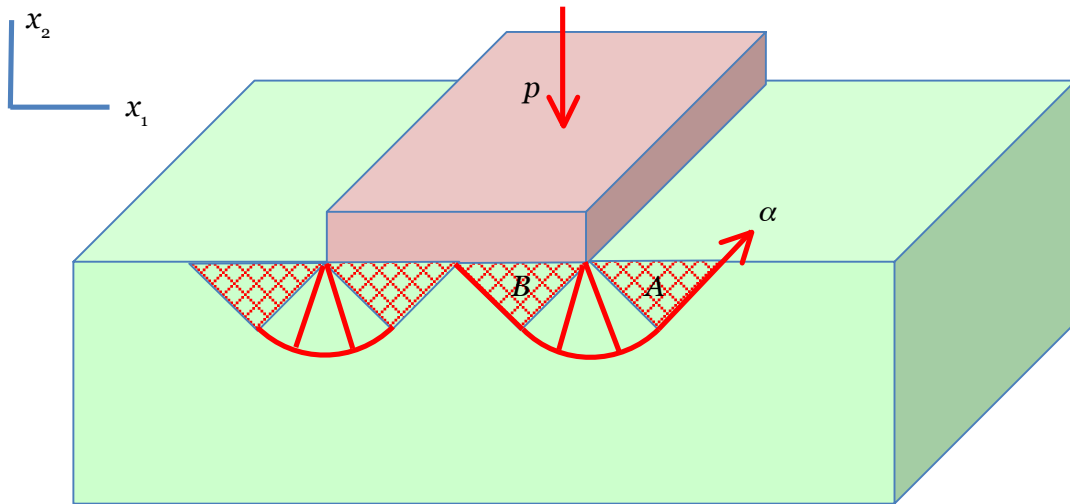
$$\sigma_{11} = \sigma_m - k \sin 2\phi = -\pi k,$$

$$\sigma_{22} = \sigma_m + k \sin 2\phi = -(2 + \pi)k.$$

The pressure needed to punch into the substrate is

$$p = (2 + \pi)k = \frac{2 + \pi}{\sqrt{3}}Y \approx 3Y$$

The stress needed for an indenter to press into the substrate is known as the *hardness*. Indentation test is a convenient method to determine the yield stress of metals.



Slip lines around cracks (Rice 1968; Hutchinson 1979).

- Small-scale yielding. The plastic zone is small compared to the sample size. The slip-line field around the crack tip is similar to that of indentation.
- A sample with two edge cracks
- A sample with a center crack. When the plastic zone is small, the mean stress is high. When the plastic zone is large, the mean stress is low. We can guess the two types of slip-line fields, but the transition between one to the other requires numerical simulation.
- This transition affects fracture energy measurement. The measured “fracture energy” depends on the size and type of the specimen.
- An edge crack in a sample under tension
- An edge crack in a sample under bending.

Equations that govern the velocity field. Inspect two equations involving the velocities:

$$\frac{\partial v_1}{\partial x_1} = \lambda \frac{\sigma_{11} - \sigma_{22}}{2},$$

$$\frac{\partial v_2}{\partial x_2} = \lambda \frac{\sigma_{22} - \sigma_{11}}{2}.$$

The components of a vector relative to two sets of base vectors are related as

$$v_1 = v_\alpha \cos \phi - v_\beta \sin \phi,$$

$$v_2 = v_\alpha \sin \phi + v_\beta \cos \phi.$$

A combination of the two sets of equations gives that

$$\cos \phi \frac{\partial v_\alpha}{\partial x_1} - v_\alpha \sin \phi \frac{\partial \phi}{\partial x_1} - \sin \phi \frac{\partial v_\beta}{\partial x_1} - v_\beta \cos \phi \frac{\partial \phi}{\partial x_1} = \lambda \frac{\sigma_{11} - \sigma_{22}}{2},$$

$$\sin \phi \frac{\partial v_\alpha}{\partial x_2} + v_\alpha \cos \phi \frac{\partial \phi}{\partial x_2} + \cos \phi \frac{\partial v_\beta}{\partial x_2} - v_\beta \sin \phi \frac{\partial \phi}{\partial x_2} = \lambda \frac{\sigma_{22} - \sigma_{11}}{2}.$$

When the coordinate (x_1, x_2) coincide locally with the orientation of slip, $\sigma_{11} - \sigma_{22} = 0$ and $\phi = 0$. The slip lines do not stretch. The two equations reduce to

$$\frac{\partial v_\alpha}{\partial s_\alpha} - v_\beta \frac{\partial \phi}{\partial s_\alpha} = 0,$$

$$v_\alpha \frac{\partial \phi}{\partial s_\beta} + \frac{\partial v_\beta}{\partial s_\beta} = 0.$$

These expressions are used to determine velocity field once the slip-line field is known.

Yield Function and Flow Potential

The Mises yield condition and the Levy-Mises flow rule are special cases of a general model.

Viscous fluid. Consider a material in a general state of stress, σ_{ij} , and rate of deformation D_{ij} . A relation between the state of stress and the rate of deformation defines a model of viscosity. Write $F(\sigma_{11}, \sigma_{22}, \dots, \sigma_{12})$ as a function that maps the state of stress to a scalar. We will write the function in shorthand, $F(\sigma)$. We define a model of viscosity by prescribing a function $F(\sigma)$, such that the rate of deformation is the gradient of the function:

$$D_{ij} = \frac{\partial F(\sigma)}{\partial \sigma_{ij}}.$$

We further assume that $F(\sigma)$ is a convex function. We have learned that this model of viscosity has many nice theoretical properties. The function $F(\sigma)$ is known the flow potential, the creep potential, or the plastic potential. The flow potential has the dimension of energy per unit time and per unit volume.

Viscoplastic fluid. We have considered the viscoplasticity. Here we summarize the idea in general terms. Let k be the yield stress of the fluid in shear. All states of stress form a linear space, the stress space. Each point in the space represents a state of stress. All the states stress that keep the fluid rigid forms a region in the stress space. The states of stress outside this region cause the fluid to flow. The boundary of the region is called the *yield surface*. Let $f(\sigma)$ be a function that maps a state of stress to a scalar, such that the yield surface is the level set of the function:

$$f(\sigma) = k^2.$$

We call $f(\sigma)$ the *yield function*. As defined here, the yield function has the dimension of stress squared. When $f(\sigma) < k^2$, the fluid is rigid, $D_{ij} = 0$. When $f(\sigma) > k^2$, the fluid flows,

$$D_{ij} = \frac{\partial f(\sigma)}{\partial \sigma_{ij}}.$$

The yield function $f(\sigma)$ and the flow potential $F(\sigma)$ are two distinct functions, but they need to satisfy the condition of continuity. For any state of

stress on the yield surface, $f(\sigma) = k^2$, the rate of deformation must vanish, $\partial F(\sigma) / \partial \sigma_{ij} = 0$.

Rigid-plastic fluid. We now generalize the rigid-plastic model to multiaxial state. Let $f(\sigma)$ be the yield function, and $F(\sigma)$ be the flow potential (also known as plastic potential). The yield condition is

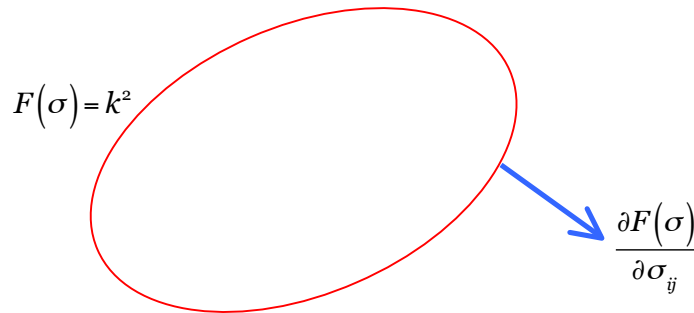
$$f(\sigma) = k^2.$$

For a rigid-plastic fluid, the state of stress never goes outside the yield surface. When the state of stress is inside the yield surface, $f(\sigma) < k^2$, the fluid is rigid, $D_{ij} = 0$. When the state of stress is on the yield surface, $f(\sigma) = k^2$, the fluid flows according to the flow rule:

$$D_{ij} = \lambda \frac{\partial F(\sigma)}{\partial \sigma_{ij}}.$$

The pre-factor λ is a positive scalar. The rigid-plastic model sets the rate of deformation in the direction normal to the level set of the flow potential, but does not specifies the magnitude of the rate of deformation.

The yield function $f(\sigma)$ and the flow potential $F(\sigma)$ are two distinct functions. The model has nice theoretical properties if we make the yield function and the flow potential identical. Such a model said to specify *associated* yield condition and flow rule. In this case, the rate of deformation is normal to the yield surface.



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