SET

Mathematics is a living language. We use it to describe the world—rivers, stars, life, engines, computers, economy, ecology, and friendship on Facebook. Every user of a living language is a creative user. We tell new stories and make up new phrases. Mathematics is too important to leave to mathematicians, just as English is too important to leave to linguists.

We learn a living language by using it. We listen, speak, read, and write. Still, books of vocabulary and syntax help. Many people derive pleasure in reading the little book by Strunk and White, *The Elements of Style*.

Here we describe the language of sets. This topic contributes much to the vocabulary and syntax of mathematics. The Internet may have replaced books of vocabulary and syntax of English and mathematics. Wikipedia contains fine articles on topics treated here, and more. The notes below serve as a guide and a reminder.

The notes consist of two types of sections: the mathematical and the applied. Each mathematical section builds on a small number of new terms, and follows a common sequence: define the new terms, illustrate the new terms with examples, derive consequences of the new terms, and describe the interplay between the new terms and old ones.

Each applied section, marked with *, describes an application of mathematics. You may skip the applied sections without losing the continuity of the mathematical sections.

Set and Element

To belong or not to belong. A collection of things is called a *set*. Each thing in the collection is called an *element*.

When a thing a is an element in a set S, we say that a belongs to S and write

$$a \in S .$$

The symbol

$$\in$$

appears between an element and a set, and is read "belongs to", or "is an element in".

When a mathematical expression appears in a separate line, its relation to the text above and below still obeys the usual rules of grammar. In the previous paragraph, the expression $a \in S$ in a separate line appears at the end of a sentence, and we put a period after the expression. By contrast, the expression $\in$ in a separate line appears in the middle of a sentence, and we do not put a period after the expression.

When an English expression, such as "a belongs to S" and "belongs to", appears in the middle of a sentence, we put the expression between quotation marks. However, we do not put a mathematical expression, such as $a \in S$ and $\in$, between quotation marks when there is no danger for confusion.

The expression $a \in S$ has the three parts of a sentence: subject, verb, and object. The expression is a complete sentence. Some writers use the symbol $\in$ both as a verb and as a preposition. We will not be so permissive. We use the symbol $\in$ as the verb "belongs to", not as the preposition "in". We write the phrase "a in S" in English, and do not translate the phrase to $a \in S$. We always regard $a \in S$ as a complete sentence. Be this convention, the phrase "for an element $a \in S$" is grammatically wrong. Instead, we write "for an element a in S".

When a thing b is not an element in the set S, we say that b does not belong to S and write

$$b \notin S .$$

The expression $b \notin S$ is also a complete sentence.

For example, let P be the set of all prime numbers. We translate the sentence "2 is a prime number" as "$2 \in P$", and translate the sentence "6 is not a prime number" as "$6 \notin P$".

Synonyms. The notion of sets permeates English. We will use the words "set", "collection", "suit", "aggregate", "family", and "class" interchangeably, but prefer the shortest word "set".

The notion of elements also permeates English. We will use the words "element" and "member" interchangeably. On occasions, we will also use words "point", "object", "thing", "item", "part", and "individual".

The flexibility in usage is nice when we speak of "a family of sets", and of "a member in a family". Compare these phrases to "a set of sets", which sounds confusing, and to "an element in a family", which sounds sinister.

Words for sets of particular things. We often use special words for sets of particular things. Examples include a school of fish, a flock of birds, a pack of wolves, a can of worms, a group of people, a library of books, a team of researchers, a stack of wood, a pile of paper, a wad of cash, a set of notes, a set of silverware, a chain of events, a bundle of joys, and a bunch of idiots.

The same set. When two sets A and B contain the same elements, we say that the two sets are "equal", or "the same", or "identical", and write

$$A = B .$$

This expression is also a complete sentence, and is read "A equals B".

Every element in a set is a distinct thing, and needs to appear only once in the set. Elements in a set can appear in any order.

Thus, a set of three things,

$$1, a, x,$$

is the same as the set

$$a, 1, x.$$

The set is also the same as the set

$$1, 1, a, x, x.$$

We should keep one of any repeated element in a set.

The number of elements in a set. A set of n elements is called an n-element set. A set that contains at least one element is called a *nonempty set*. A set of exactly one element is called a *singleton*.

The set that contains no element is called the *empty set*, and is denoted by $\varnothing$. By definition, all empty sets are identical.

We will often deal with a set of infinitely many elements. For example, we will examine moments in time, points in space, lumps of mass, amounts of energy, and degrees of temperature.

Other sets of infinitely many elements include the set of positive integers N, the set of integers Z, the set of rational numbers Q, the set of real numbers R, and the set of complex numbers C. We assume that you know what these sets are. If you do not, look them up.

Structured set. Algebra is devoted to the study of structured sets. For example, the set of positive integers has one element following another in an order. The addition of any two elements in the set gives another element in the set. The multiplication of the any element in the set gives yet another element in the set. We will describe these structures on sets later.

Unstructured set. But most sets useful to us do not have any "natural" structure. For example, the set of all nations in the world does not have any "natural" structure. We can, of course, force a structure on this set. For example, English speakers often list nations by the alphabetical order of their English names. This order makes no sense to non-English speakers.

To force a structure on a set is to impose a structure that is not inherent to the set. If the imposition confuses us, it is just a bad use of language. Alphabetical order is convenient, but incidental. We should never let this incidental structure bias us. If no structure is needed, we should simply regard the set of nations as an unstructured set.

Similarly, we regard a school of fish, a pile of wood, and a class of students as unstructured sets.

Ways to Specify a Set

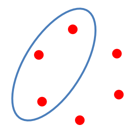
Specify a set by pointing to its elements. By specifying a set we mean specifying the elements in the set.

"These are my children," says a mother. She might point to the children themselves, or to a photo of the children.

She then snaps a photo of the dishes on the table, and posts the photo on a social network. The photo tells the world the set of dishes. A picture is worth a thousand words.

As another example, here is a set of three animals:

Specify a set by dots and a bubble. For a small number of things, we can represent each thing as a dot on a piece of paper. If we wish to differentiate the dots, we can attach a distinct word to each dot. We then draw a bubble to include things that belong to a set.



This way to specify a set has many variations. For example, if each thing represents a person, we can write the name of the person. Inside the bubble, for example, specifies a set of people in a committee.

Specify a set by listing the names of its elements. Manipulating names of things are much simpler than manipulating the real things. We often specify a set by listing the names of its elements between braces, $\{\ \}$. For example, we can specify the set of three animals as

$$\{\text{cat, dog, rabbit}\}.$$

The notation appears in the above sentence as a noun phrase, and is read "the set of cat, dog, and rabbit".

The notation

$$\{\text{Daniel, Michael}\}$$

denotes a set of two elements, two boys named Daniel and Michael.

The notation

$$\{\text{Einstein, Beethoven}\}$$

denotes another set of two elements, the two hamsters kept by the two boys.

English sentences usually mention sets without putting braces around elements. For example, we say, "We gathered over wine, cheese, and cracker." The sentence mentions two sets: "we"—a set of people; and "wine, cheese, and cracker"—a set of three things.

Specify a set by a phrase. Listing all elements in a set is tedious if the set has many elements, and is impossible if the set has infinitely many elements.

The notation

$$\{\text{Alabama, Alaska,..., Wyoming}\}$$

denotes the set of 50 states in the United States. In writing this way, we assume that the reader knows what we mean. If we do not wish to make this assumption, we may specify the set by English. We say, "the set of the 50 states in the United

States ". This English phrase does not tell us the elements of the set directly, but prompts us to find them. In this sense, the English phrase "specifies" the set.

The notation

$$\{\text{Afghanistan,..., Zimbabwe}\}$$

is probably too unclear to be useful. Rather, we say, "the set of all countries in the world." We go online to learn about their names, locations, populations, etc. My last search shows that the world has a total of 257 countries.

The notation

$$\{1,2,3,...\}$$

denotes the set of positive integers. This set has an infinitely many elements. Alternatively, we denote the set of positive integers by *N*. We know this set by heart since our childhood.

Listing elements vs. stating a property. The phrase that specifies a set is called a *property*.

The property "the states in the United States that begin with letter A" defines the set

$$\{\text{Alabama, Alaska, Arizona, Arkansas}\}.$$

The property "the states bordering Massachusetts" defines the set

$$\{\text{New York, Vermont, New Hampshire, Rhode Island, Connecticut}\}.$$

Given a property, we can specify a set. Conversely, given a set, we often wish to find a property that specifies the set. This wish is instinctive, but unnecessary. What is the property that specifies the set

$$\{\text{Massachusetts, California, New Jersey}\}?$$

We can, of course, come up with a property that specifies this set. For me, the set is specified by the following property: the states in which I have held long-term jobs. But this property is unknown to other people. Nonetheless the set is well defined once we list the three states.

The same is true of the set

$$\{\text{rock, paper, scissors}\}.$$

In some cultures the set represents a game. Paper covers rock, rock crushes scissors, and scissors cut paper. But the set makes no sense in other cultures. Nonetheless, the list of three things is a set.

Consider a set of three elements:

$$\{\text{egg, pen, car}\}.$$

Why do we even call this a set? The three things do not seem to belong to one another. But the three things fit the mathematical notion of set: any collection of any things is a set.

We will regard elements as primary. Once we specify the elements, we have specified a set. Whether we specify the elements by a picture, by symbols, or by a property makes no difference. Besides, the same set can be defined by many properties.

Set-building notation. That said, a set becomes useful typically because we specify its elements by a property. When the membership of a set S is determined by a property p, we write

$$S = \{x | x \text{ has property } p\},$$

which is read "S is the set of x such that x has the property p". The expression $\{x | x \text{ has the property } p\}$ is called the set-building notation.

For example, we write

$$C = \{x | x \text{ is a country in the world}\},$$

which is read "C is the set of x such that x is a country in the world". The symbol x stands for an arbitrary element in the set C, and is called a *variable*. Afghanistan is a particular element in the set C, and is called a *value* of the variable. The above definition does not list the countries in the world, but we know where to find them. In this sense, the above definition specifies the set C.

As another example, we write

$$S = \{x | x \text{ is a person who loves Einstein}\}.$$

This definition confuses us. Depending on the meaning of the word "Einstein", the physicist or the hamster, the set S may denote a large number of people in the world, or just the two boys. Mathematicians can be as confusing as anyone. If you are confused, ask for clarification, or read from the context.

Union and Intersection

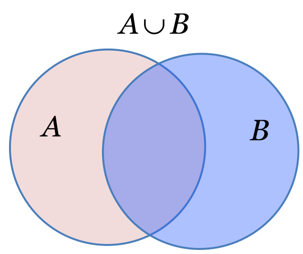
Much of mathematics is about creating new sets from old ones. Let us begin with examples that can be represented by the Venn diagrams, which you have learned in high schools.

Union of two sets. The *union* of two sets A and B, written as $A \cup B$, is the set of all elements that belong to either A or B.

Using the set-building notation, we write

$$A \cup B = \{x | x \text{ is in } A \text{ or } B\}.$$

The expression "$A \cup B$" is read "the union of A and B", and is a noun phrase. The symbol "$\cup$" appears between two sets, but is read "the union of". We will never use the symbol as a verb.



Here are two examples of unions:

$$\{\text{Lisa, Eric}\} \cup \{\text{Daniel, Michael}\} = \{\text{Lisa, Eric, Daniel, Michael}\},$$

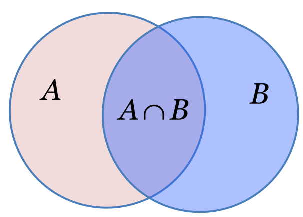
$$\{b,c,1,2\} \cup \{b,c,d,e\} = \{b,c,1,2,d,e\}.$$

Intersection of two sets. The *intersection* of two sets A and B, written as $A \cap B$, is the set of all elements that belong to both A and B.

Using the set-building notation, we write

$$A \cap B = \{x | x \text{ is in } A \text{ and } B\}.$$

The expression "$A \cap B$" is read "the intersection of A and B", and is also a noun phrase.



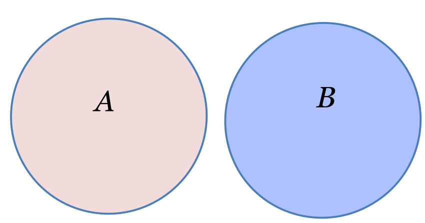
Here are two examples of intersections:

$$\{\text{Lisa, Eric, Lucas}\} \cap \{\text{Daniel, Michael, Lucus}\} = \{\text{Lucus}\},$$

$$\{b,c,1,2\} \cap \{b,c,d,e\} = \{b,c\}.$$

Disjoint sets. Two sets A and B are called *disjoint* when they have no element in common—that is, when the intersection of the two sets is the empty set:

$$A \cap B = \varnothing.$$

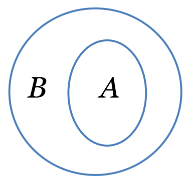


For example, the two sets $\{\text{Lisa, Eric}\}$ and $\{\text{Daniel, Michael}\}$ are disjoint:

$$\{\text{Lisa, Eric}\} \cap \{\text{Daniel, Michael}\} = \varnothing .$$

Subset

Subset. If every element in a set A is also an element in a set B, we say that A is a *subset* of B. If in addition there is at least one element in B but not in A, we say that A is a *proper subset* of B.



If A is a subset of B, we write

$$A \subset B .$$

Equivalently, we say that B contains A and write

$$B \supset A .$$

The symbol $\subset$ appears between two sets, and is read "is a subset of", or "is contained by". The symbol $\supset$ is read "contains". We use both symbols as verbs.

Examples. For any set S, the empty set is a subset of S, and the set S is itself a subset of S.

For any two sets A and B, both A and B are subsets of $A \cup B$, and the set $A \cap B$ is a subset of A and a subset of B.

The set $\{\text{Eric}\}$ is a subset of the set $\{\text{Eric, Daniel, Michael}\}$.

The set $\{2,5\}$ is a subset of the set $\{1,2,3,4,5,6\}$.

The basic sets of numbers form a string:

$$N \subset Z \subset Q \subset R \subset C .$$

The set of prime numbers P is a subset of the set of positive integers N:

$$P \subset N .$$

Specify a subset by a property. Let us return to the property (phrase)

"the states in the United States that begin with letter A".

We now parse this phrase into two parts, and examine their syntactic roles.

The part "the states in the United States" defines the whole set S of all the 50 states in the United States:

$$S = \{\text{Alabama, Alaska,..., Wyoming}\} .$$

The part "that begins with letter A" is a modifier that restricts the choices of the states, and defines a set A of four elements:

$$A = \{\text{Alabama, Alaska, Arizona, Arkansas}\} .$$

The set A is a subset of S:

$$A \subset S .$$

Thus, a property that specifies a subset should contain two parts: one part specifies the whole set, and the other part specifies the subset.

A computer stores the addresses of manyf people. These addresses form a set S. We can search for the addresses J of people by the name John Doe. The set J is a subset of the set S.

Every set is a subset of some larger set. If the part that specifies the whole set is missing from the property, look for the part in the context. For example, the phrase "ivy league" specifies a set of eight universities:

$$\{\text{Brown, Columbia, Cornell, Dartmouth, Harvard, Penn, Princeton, Yale}\}.$$

We may regard this set as a subset of the set of all universities.

Power Set

Power set. The *power set* of a set S is the collection of all subsets of S. Thus, a power set of S is a set, each element of which is a subset of S. Each subset of the power set is called *a family of subsets* of S.

For example, let S be a two-element set:

$$S = \{a, b\}.$$

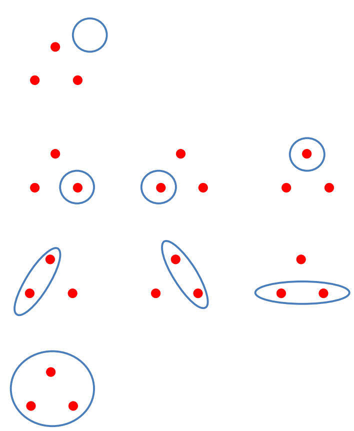
This set has a total of four subsets:

$$\varnothing, \{a\}, \{b\}, \{a, b\}.$$

The four subsets constitute the power set of S.

Dots and bubbles. For a set S of a small number of elements, we can represent each element of S by a dot, and represent each subset of S by a bubble.

For example, the power set of a three-element set S contains a total of eight subsets:



How many subsets can we create from a set? That is, how large is the power set? Now let S be an n-element set:

$$S=\{s_1,...,s_n\}.$$

The power set of S is the collection of all subsets of S:

$$\varnothing,\{s_1\},...,\{s_n\},\{s_1,s_2\},...,\{s_{n-1},s_n\},...,\{s_1,...,s_n\}.$$

In creating a subset of S, we need to decide whether or not to include each of the n elements, $s_1,...,s_n$. Consequently, the n independent binary decisions create a total of 2^n subsets.

Partition

Partition and part. A *partition* P of a set S is a family of nonempty subsets of S, which are disjoint, and whose union is S.

Thus, a partition P of a set S is itself a set, and the definition specifies the set P by a property. The property itself has several distinct ingredients. The above definition packs these ingredients into a single sentence. But we can always unpack the sentence and list the ingredients one by one:

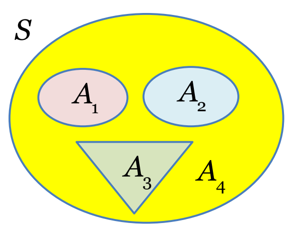
(1) P is a family of sets $A_1,...,A_m$: $P=\{A_1,...,A_m\}$.

(2) Every member in P is a subset of S: $A_i \subset S$.

(3) Every member in P is nonempty: $A_i \neq \varnothing$.

(4) Every pair of members in P are disjoint: $A_i \cap A_j = \varnothing$.

(5) The union of all members in P is S: $A_1 \cup ... \cup A_m = S$.



Each member in the family P is a subset of S, and is called a *part* of P. Each element in S belongs to one and only one part in P. A partition P of a set S is a subset of the power set of S.

Examples. Let S be a set of five people:

$$S=\{\text{Lisa, Eric, Daniel, Michael, April}\}.$$

The partition

$$\{\{\text{Lisa, April}\},\{\text{Eric, Daniel, Michael}\}\}$$

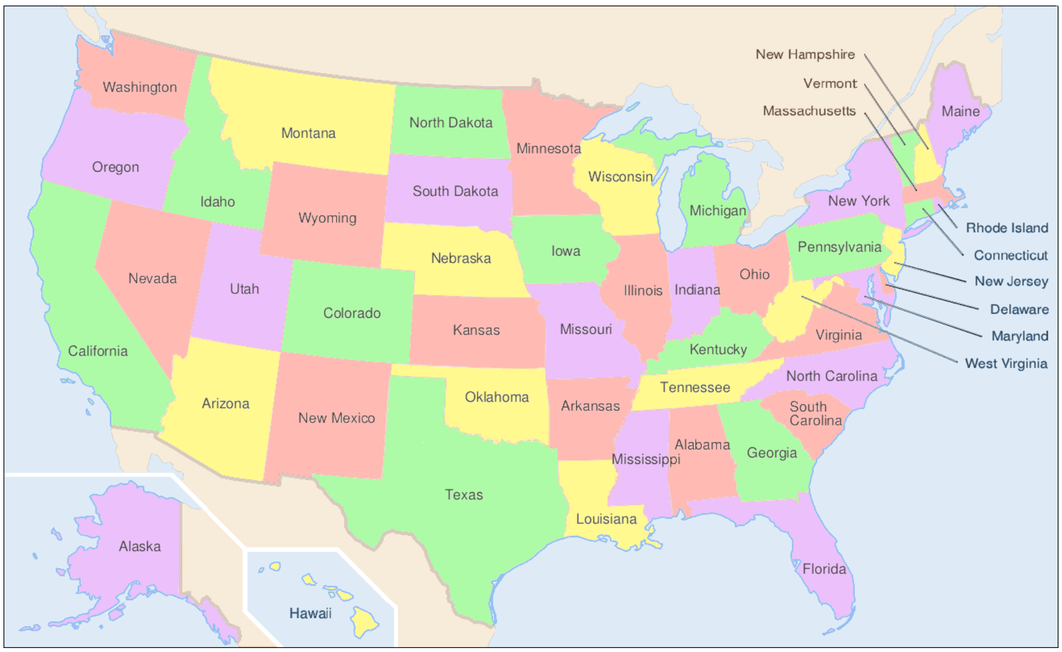
divides the set S into the subsets of girls and boys. The partition

$$\{\{\text{Lisa, Eric}\},\{\text{Daniel, Michael}\},\{\text{April}\}\}$$

divides the set S into three subsets, each of which consists of children in a family: Lisa and Eric are sister and brother, Daniel and Michael are brothers, and April is a single child. Of course, we can create many more partitions of the set S. Indeed, we can create a partition without stating any property that specifies the partition.

The students taking a course form a set S. Each student receives one and only one grade. Five grades are available: A, B, C, D, and F. The students receive the same grade form a subset of S. Thus, the grades give a partition of S.

The United States of America is divided into 50 states. This division constitutes a partition of the whole set, the United States America.



The set of integers can be partitioned into the set of odd numbers and the set of even numbers.

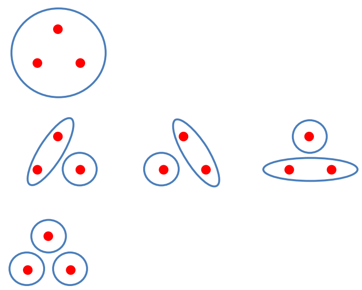
The set of integers can also be partitioned into the set of prime numbers and the set of composite numbers.

The set of real numbers can be partitioned into the set of rational numbers and the set of irrational numbers.

How many partitions can we create from a set? A three-element set $\{x, y, z\}$ has a total of five partitions:

$$\{\{x,y,z\}\}$$
$$\{\{x\},\{y,z\}\}$$
$$\{\{y\},\{z,x\}\}$$
$$\{\{z\},\{x,y\}\}$$
$$\{\{x\},\{y\},\{z\}\}$$

We can also represent the three-element set by three dots, and represent each subset by a bubble. The five partitions look like this:



The number of partitions of an *n*-element set is called the Bell number B_n. The first several Bell numbers are $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, and $B_5 = 52$.

Coarsen and refine a partition. Each member in *P* is a subset of *S* and is called a part of *P*. The union of several parts is a larger subset of *S*.

A partition *P* of a set *S* is a set in its own right. We can create a partition of *P*. The family of the new subsets also forms a partition of *S*. This new partition is said to *coarsen* the partition *P*.

Each part of *P* is a set in its own right. We can partition the part into smaller subsets of *S*. The new family of subsets forms a partition of *S*. This new partition is said to *refine* the partition *P*.

Folder and Tag*

A person organizes his files. Let us describe methods of organizing files using the language of sets. All his files form a set, an element of which is an individual file. By organizing files, the person means that he divides the whole set of files into subsets.

Folder. In the physical world, he organizes the files using folders. He places each file in one and only one folder. Multiple copies of the same file in different folders would confuse him, particularly if he will add comments to the file repeatedly.

In this filing system, each folder contains a nonempty subset of files, and each file is in one and only one folder. Thus, all folders constitute a partition of the whole set of files.

The person can combine existing folders to coarsen the partition, and create subfolders to refine the partition.

This filing system by folders is so familiar to us that we keep using it even for files on computers. But filing by folders has been a constant source of frustration. "Did I put that file in the folder of mathematics, physics, or mathematical physics?"

Tag. A different filing system becomes practical on computers. The person can give each file multiple tags. Files with the same tag belong to a subset. Each file can belong to multiple subsets. Thus, he can give that file three tags: mathematics, physics, and mathematical physics. No matter which tag he clicks, he will find the same file, and can edit the file repeatedly.

This filing system by tags has become popular for blog posts, but not so much for people organizing files on their own computers.

A Long Molecule in Water*

Here is an example of using the language of sets to describe a physical phenomenon.

Microstates of an isolated system. Consider a long molecule in water. We say that the long molecule and water molecules form a system. The system is said to be an *isolated system* when we block all interactions between the system and the rest of the world.

Even in the isolated system, the long molecule and water molecules jiggle ceaselessly. That is, the isolated system flips among a huge number of detailed molecular arrangements, which we call microstates. All the microstates constitute a set S, which we call the *sample space* of the isolated system.

Just as we use the phrase "all states of the Unites States" to specify the set 50 states, we use the phrase "a piece of isolated cheese" to specify a set of a huge number of microstates. Thus, an isolated system is synonymous to a set of microstates, or a sample space.

Configurations of an isolated system. The individual microstates are too numerous to interest us. We look for less detailed ways to describe the isolated system. Let us consider an example.

The long molecule can be in two *configurations*: a loop or a chain. The long molecule is a loop when its two ends are bonded together, and is a chain is when its two ends are separated.

The two configurations—loop and chain—are subsets of *S*. Each configuration consists of many microstates. For example, both a straight chain and a wiggled chain belong to the configuration of chain. Besides, the water molecules in the surrounding ceaselessly move.

The two subsets are disjoint, and their union is *S*. Thus, the two configurations constitute a partition of *S*.

An experiment defines a partition of the sample space. The two configurations can be differentiated by an experiment. We can synthesize the long molecule so that when its two ends meet it produces a photon.

The light-detecting experiment partitions *S* into the two subsets: one associated with a chain, and the other associated with a loop.

Translation. In an application, words in mathematics acquire specific meanings, and are often translated into words specific to the application. For this application, here is a dictionary:

- Element: microstate of an isolated system
- Set: sample space, a set of microstates, an isolated system
- Subset: a subset of microstates, a configuration
- Partition: loop and chain

Thermodynamics and statistical mechanics. The above example can be generalized to any isolated system, be it a piece of isolated cheese, or a bottle of isolated wine. We design experiments that divide the set of microstates into various partitions. The subject is known as thermodynamics or statistical mechanics.

Randomness*

Toss a coin, and it falls either head or tail. Roll a die, and it settles with one of the six faces. The theory of probability generalizes these games using the language of sets.

Sample and sample space. Tossing a coin is called an *experiment*, and each toss is called a *trial* of the experiment. Each toss gives one of two *outcomes*, head and tail.

Rolling a die is another experiment, and each roll is called a trial of the experiment. Each roll gives one of the six faces, commonly labeled with numbers 1, 2, 3, 4, 5, and 6.

Consider an experiment, each trial of which gives one of n outcomes, $s_1,...,s_n$. Each outcome is called a *sample*, and the set of all outcomes is called the *sample space*. We designate the sample space by S and write

$$S = \left\{ s_1,...,s_n \right\}.$$

Each outcome of a trial of the experiment is an element in the sample space.

Sample space is an unstructured set. In labeling the samples with numbers 1,...n, we merely wish to differentiate the outcomes, but do not wish to imply preference for any order on the sample space.

For example, we can designate the sample space of the experiment "tossing a coin" by either $\left\{ \text{head, tail} \right\}$ or $\left\{ \text{tail, head} \right\}$. The two designations represent the same sample space.

The faces of a die do not have to be labeled by numbers. To guard ourselves against needless numbers, we may use letters a, b, c, d, e, and f to label the six faces. In doing so, we make it clear that neither addition nor multiplication of two faces of a die needs to have any meaning.

Labeling the faces with letters a, b, c, d, e, and f should not imply preference for an order of faces. To guard ourselves against needless orders, we may draw a different picture on each face. For instance, on the each face we may draw a figure, such as

ant, bee, cat, dog, eel, and fly.

Unless other wise specified, we regard a sample space as an unstructured set.

Event. Any subset of the sample space of an experiment is called an *event*. A trial of the experiment is said to give the event if the trial gives any one of the samples in the subset.

Examples. The empty set is an event, which no trial of the experiment will give. The sample space is another event, which every trial of the experiment will give.

For a die of faces $\left\{ a,b,c,d,e,f \right\}$, the subset $\left\{ b,d,f \right\}$ corresponds to an event. The event is said to occur if a roll of the die results in either face b, or face d, or face f.

Now consider a die of faces $\{1,2,3,4,5,6\}$. The event that a roll of the die gives an even number is $\{2,4,6\}$. The event that a roll of the die gives a number larger than 4 is $\{5,6\}$.

Specify an event by outcomes or by a property. Any subset of the sample space is an event. For example, $\{1,4\}$ is an event of rolling a die, even though we are unsure if we have a property (a phrase) that defines the event. The event is specified by the two outcomes, and simply means that a roll of the die gives either 1 or 4.

That said, usually an event becomes useful when we can specify it by a property. Indeed, many problems in the theory of probability boil down to translating a property that defines an event to a subset of the sample space.

Event space. All events of an experiment constitute a set, which we call the *event space*. We denote all events of an experiment by $A_1,...,A_m$, and the event space by E. Thus,

$$E = \{A_1,...,A_m\}.$$

For example, the sample space of tossing a coin, $\{\text{head, tail}\}$, has a total of four subsets:

$$\varnothing, \{\text{head}\}, \{\text{tail}\}, \{\text{head, tail}\}$$

They are the events of the experiment. The four events together constitute the event space.

An experiment of n samples can form a total of 2^n events. These events constitute the event space of the experiment.

Partition of sample space. The events $\{b,d,f\}$ and $\{a,c,e\}$ are disjoint subsets of the sample space of a die, and their union is the sample space. Consequent, the two events constitute a partition of the sample space. The sample space has many partitions. Here is another partition of the sample space: $\{a,b\}$, $\{c\}$, and $\{d,e,f\}$.

Study an experiment at a suitable level of detail. The sample space is a special event, so is every sample. Also, we can regard an event as a sample space: a subset may be considered as a set in its own right; we speak of a subset merely to indicate that we have a larger set in the back of our minds.

Furthermore, we can divide a sample space into a family of disjoint events, and then regard each event as a sample.

This last change of view is particularly important because individual outcomes of an experiment are often too numerous to interest us; we would rather focus on some aggregates of outcomes.

When tossing a coin, a gambler is interested in whether the outcome is a head or a tail. Thus, the gambler regards "tossing a coin" as an experiment of sample space $\{\text{head}, \text{tail}\}$. Of course, the coin is made of atoms, each atom is made of electrons, protons and neutrons, and each neutron is made of... Associated with a head, the coin has numerous quantum states. All quantum states constitute a sample space, and the set $\{\text{head}, \text{tail}\}$ is merely a partition of the sample space. But this mode of description is too detailed to interest the gambler, unless she happens to be a physicist.

Translation. We have just described randomness using the language of sets. Here is a dictionary:

- Element: a sample, an outcome of an experiment
- Set: sample space
- Subset: event
- Power set of the sample space: event space
- Partition: a family of nonempty events, which are disjoint, and whose union is the sample space