STRAIN HARDENING

Rheology. Rheology is the science of deformation. This science poses a question for every material: given a history of stress, $\sigma(t)$, how do we find the history of strain, $\varepsilon(t)$?

We can certainly apply the history of stress to the material and record the history of strain. We can also use computers to simulate the movements of electrons and atoms and molecules. These brute-force approaches, however, do not work by themselves. The stress has six components and can each undergo countless histories. There are other factors and their histories to consider: temperature, electric field, humidity, pH, etc. There are so many materials: metals, glasses, rubbers, liquids, toothpastes, and chewing gums. They all behave differently. How many experiments and simulations do we have to run?

In various courses we have been learning a hybrid approach. Given a material, we paly with it and watch it deform. We relate histories of stress and histories of strain by constructing a phenomenological model. We determine parameters in the model by running experiments. We try to understand the model by thinking about electrons and atoms and molecules, sometimes aided by computers. The hybrid approach dates back at least to Hooke's law of elasticity and Newton's law of viscosity. More recent examples include various models of thermoelasticity, viscoelasticity, large-strain elasticity, plasticity, poroelasticity, electrorehology, and chemorheology.

This course focuses on plasticity, but we also compare plasticity with other types of rheological behavior. Furthermore, we examine how the rheology of materials affects phenomena of inhomogeneous deformation, such as necking, cavitation, creasing, and shear localization. In a separate course, we have examined how the rheology of materials affects fracture.

In this lecture we will construct the elastic, isotropic hardening model for a metal. The model achieves something extraordinary: it uses the experimental record of a single history (i.e., the tensile stress-strain curve) to describe all histories of loading and unloading, tension and compression. Here we will consider a rod of metal under axial force, but later in the course we will generalize the model to multiaxial loading.

Perfect plasticity vs. strain hardening. In previous lectures we have focused on perfect plasticity. This model assumes that yield strength of a metal is constant, independent of the amount of deformation. The stress-strain diagram consists of three parts:

- A horizontal line at the positive yield strength represents plastic flow in tension.
- A horizontal line at the negative yield strength represents plastic flow in compression.
- Infinitely many straight lines of the slope of Young's modulus represent elastic loading and unloading.

We now construct a model to capture a salient feature of metal plasticity: strain hardening. As a metal undergoes plastic deformation, the yield strength keeps increasing.

Strain hardening in a metal originates from the multiplication of dislocations (Taylor 1934). Plastic deformation in the metal results from the movement of dislocations. Dislocations, however, impede the motion of one another. As plastic deformation generates more and more dislocations, the metal becomes harder and harder.

By contrast, the elastic modulus of the metal is unaffected by deformation. Dislocations move and multiply, atoms change neighbors, but the elasticity of the bonds between the atoms remain unchanged.

Here we will not study the atomistic origin of strain hardening, but will construct a phenomenological model to describe strain hardening, and will apply the model to analyze macroscopic phenomena.

Stress-strain diagram. We apply an axial force to a rod of a metal, and record the change in its length. We then convert the experimental data to a stress-strain diagram. Define the true stress by $\sigma = F/a$, where F is the force, and a is the cross-sectional area of the rod in the current state. Define the natural strain ε using $d\varepsilon = dl/l$, where l is the length of the rod in the current state, and dl is the increment of the length of the rod.

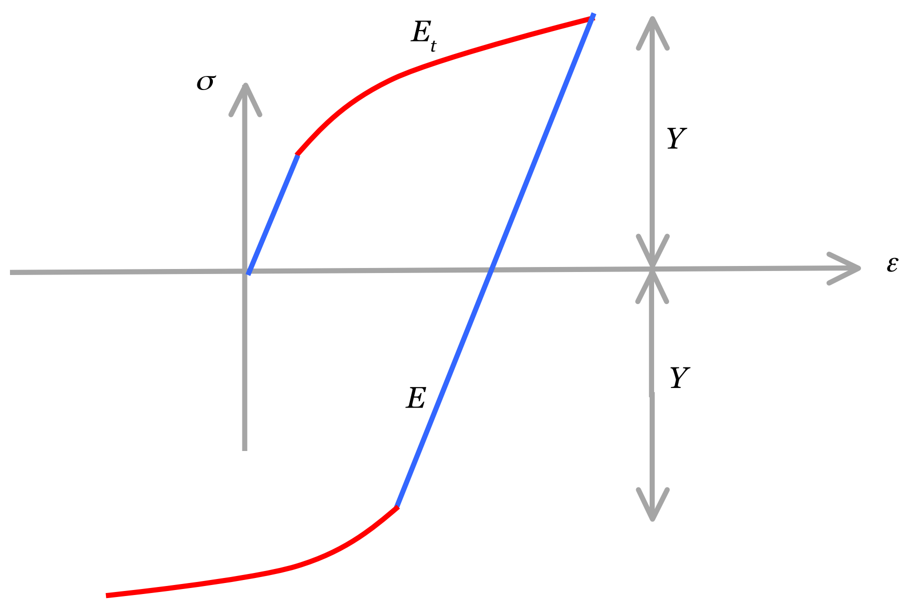
When the stress is small, the metal deforms elastically, and the stress and strain increase along a straight line of the slope of Young's modulus. Upon unloading, the stress and strain decrease along the same straight line.

We next increase the stress again, beyond the initial yield strength, so that the stress-strain curve bends, and the metal deforms plastically. This statement means several important things.

First, from a point on the bent curve, upon unloading, the stress and strain do not trace back along the bent curve; rather, they reduce along a straight line of the slope of Young's modulus. Young's modulus of the metal is constant, independent of the amount of deformation.

Second, upon reloading, the metal initially follows the straight line of elasticity, and then bends at the previous point on the curve. That is, the previous plastic flow has hardened the metal and increased the yield strength to a new level. The bent stress-strain curve in tension tells us how the yield strength increases with the tensile strain.

Third, upon unloading to zero stress, we can compress the metal. When the magnitude of the compressive stress is small, the metal follows the straight line of the slope of Young's modulus. At some level of the compressive stress, the metal yields in compression, and the stress-strain curve bends. As an idealization, we assume that strain hardening is isotropic: the metal yields in at the current yield strength established by prior deformation, and the bent stress-strain curve in compression is the same as that in tension.



Elastic, isotropic hardening model. Let us return to the main question in rheology. Given a metal and a history of stress, $\sigma(t)$, how can we find the history of strain, $\varepsilon(t)$?

The elastic, isotropic hardening model answers this question by making idealizations mentioned above. The model also needs an input from experiment. We need to subject the metal to a tensile test without unloading, and record the tensile stress-strain curve $\sigma(\varepsilon)$. This function increases monotonically for the strain-hardening metal. The tensile test records the history of strain of the metal subject to a single history of stress: the monotonic, increasing, tensile stress. The elastic, isotropic hardening model then predicts the history of strain for the metal subject to any history of stress, unloading and reloading, in tension and in compression. Let us write this model as a list of rules that a computer can follow.

After the tensile stress-strain curve bends, its slope reduces greatly. The slope of the stress-strain curve defines the tangent modulus E_t:

$$d\sigma = E_t d\varepsilon .$$

The tangent modulus is a function of the tensile strain, $E_t = d\sigma(\varepsilon)/d\varepsilon$. Due to strain hardening, the tensile stress increases with the tensile strain, so that the tangent modulus is positive and is also a function of the tensile stress, $E_t(\sigma)$. In the tensile test, each level of stress beyond the initial yield strength represents the current yield strength. To free up the symbol σ for later use as the applied stress, we write the tangent modulus as a function of the current yield strength, $E_t(Y)$. Typically, the tangent modulus is much lower than Young's modulus, $E_t << E$.

In the elastic, isotropic hardening model, the function $E_t(Y)$ and Young's modulus E fully specify the metal. Given a history of stress, the model gives the history of strain and the history of the yield strength. At a given point in the history of stress, we know the current yield strength Y, the current stress σ, and the increment of stress $d\sigma$. The model updates the yield strength and the strain as follows.

If the current stress is in the elastic range, $|\sigma| < Y$, the metal loads and unloads along the straight line of Young's modulus. If the current stress satisfies the yield condition, $|\sigma| = Y$, and the increment of the stress causes unloading, $d|\sigma| \le 0$, the metal unloads along the straight line of Young's modulus. In both cases, the increment of the stress does not change the yield strength Y, but changes the strain by

$$d\varepsilon = \frac{d\sigma}{E}.$$

If the current stress satisfies the yield condition, $|\sigma| = Y$, and the increment of the stress causes plastic loading, $d|\sigma| > 0$, the metal loads along the bent curve. The increment of the stress changes the yield strength by $dY = |d\sigma|$, and changes the strain by

$$d\varepsilon = \frac{d\sigma}{E_t(Y)}.$$

The model achieves something extraordinary: it uses the experimental record of a single history (i.e., the tensile stress-strain curve) to describe all histories of loading and unloading, tension and compression.

Separate elastic strain and plastic strain. We adopt the assumption that the total strain is a sum of the elastic strain and plastic strain:

$$\varepsilon = \varepsilon^e + \varepsilon^p.$$

The elastic strain relates to the stress as

$$\varepsilon^e = \frac{\sigma}{E}.$$

In the stress-strain diagram, we draw a straight line of slope E to represent elastic loading and unloading. The projection of this line on the axis of strain is the elastic strain. Because the stress is usually much smaller than Young's modulus, the straight line is nearly vertical, and the elastic stain is typically less than 1%. The elastic strains in the two transverse directions are $-\nu\sigma / E$, where ν is Poisson's ratio.

If the current stress satisfies the yield condition, $|\sigma| = Y$, and the increment of the stress causes plastic loading, $d|\sigma| > 0$, the metal loads along the bent curve. The increment of stress changes the yield strength by $dY = |d\sigma|$, and changes the plastic strain by

$$d\varepsilon^p = \left(\frac{1}{E_t(Y)} - \frac{1}{E}\right)d\sigma.$$

This expression gives the increment of strain in the axial direction. The plastic strains in the two transverse directions are one half that in the axial direction.

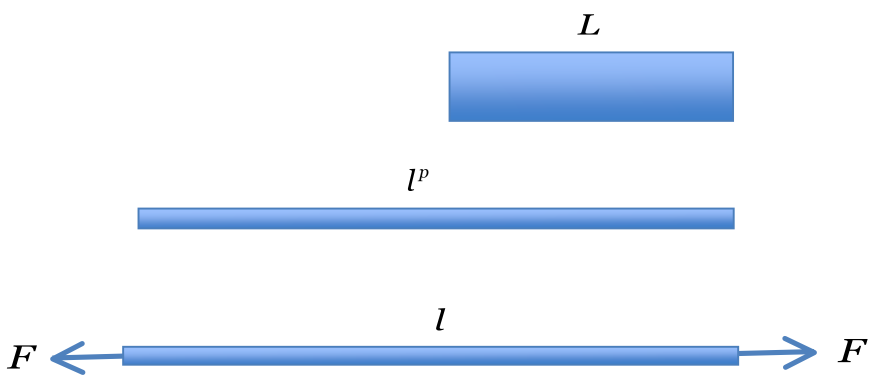
What does $\varepsilon = \varepsilon^e + \varepsilon^p$ mean? The rod is of length L in the initial stress-free state, and is of length l in the current state under stress. The ratio of the two lengths defines the stretch, $\lambda = l / L$. Starting from the current state, we remove the stress, so that the rod unloads elastically and reaches an intermediate state of length l^p. Both the initial state and the intermediate state are stress-free, but the two states differ by plastic deformation; define the plastic stretch by $\lambda^p = l^p / L$.

The intermediate state and the current state differ by an elastic deformation; define the elastic stretch by $\lambda^e = l / l^p$.

Note an identity:

$$\lambda = \lambda^e \lambda^p .$$

Recall the definition of the natural strain, $\varepsilon = \log \lambda$. Similarly, write $\varepsilon^p = \log \lambda^p$ and $\varepsilon^e = \log \lambda^e$. The expression $\lambda = \lambda^e \lambda^p$ is equivalent to $\varepsilon = \varepsilon^e + \varepsilon^p$.



Plastic flow of a metal vs. viscous flow of a liquid. A metal undergoes plastic flow as atoms change neighbors. A liquid undergoes viscous flow and molecules change neighbors. But the plastic flow of the metal differs from the viscous flow of the liquid in an important way. When a metal undergoes plastic flow, the stress depends on the amount of deformation, but is insensitive to the rate of deformation. By contrast, when a liquid undergoes viscous flow, the stress is insensitive to the amount of deformation, but depends on the rate of deformation.

We say that metals and liquids have different types of rheology. This difference must have caused a serious challenge to researchers: it segregates solid mechanicians and fluid mechanicians into different communities. They publish in different journals and go to different conferences.

Natural strain vs. rate of deformation. The natural strain commonly used in plasticity is closely related to the rate of deformation commonly used in fluid mechanics. Let l be the length of a sample in the current state. We define the natural strain ε by

$$d\varepsilon = \frac{dl}{l} .$$

We define the rate of deformation by

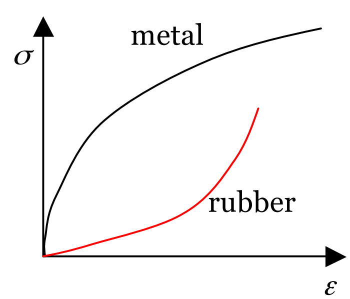
$$D = \frac{dl}{ldt}.$$

Strain-hardening in a metal vs. strain-stiffening in a rubber. Both a metal and a rubber are capable of large deformation, but they deform by entirely different microscopic mechanisms. The metal consists of atoms, and the large deformation corresponds to the atoms changing neighbors. The rubber is a three-dimensional network of corsslinked, long-chained polymers, and the large deformation corresponds to the molecules changing conformation. This difference in microscopic process causes difference in macroscopic behavior in several significant ways.

First, the large deformation in a metal is plastic, but the large deformation in a rubber is elastic. Once the atoms in the metal have change neighbors, they forget their old neighbors and do not return to their old neighbors upon unloading. The crosslinks in the rubber tie the individual long-chained molecules into a single three-dimensional network. During deformation, a small segment of one chain may slip pass a small segment of another chain, but the crosslinks will not allow chains to change neighbors. Upon unloading, the chains recover their initial conformation.

Second, the deformation in a metal can be arbitrarily large, but the deformation in a rubber has a limit, known as the extension limit.

Third, as the strain increases, the stress increases for both the metal and the rubber. We call the rising stress-strain curve strain-hardening for the metal, and strain-stiffening for the rubber. The different words—hardening and stiffening—are consistent with the common usages in plasticity and elasticity.



Fit a stress-strain curve to an expression. For a **metal** undergoing large, plastic deformation, the stress-strain curve (without unloading) is often fit to a power law:

$$\sigma = K\varepsilon^N,$$

where K and N are parameters to fit experimental data. Some representative values: N = 0.15-0.25 for aluminum, N = 0.3-0.35 for copper, N = 0.45-0.55 for stainless steel. K has the dimension of stress; it represents the true stress at strain $\varepsilon = 1$. Representative values for K are 100 MPa – 1GPa.

For a **rubber**, the stress-strain curve is often fit to a relation known as the neo-Hookean model:

$$\sigma = \mu\left[\exp(2\varepsilon) - \exp(-\varepsilon)\right].$$

Representative values for μ are 0.1 MPa – 10 MPa. The neo-Hookean model, however, does not describe the extension limit.