VECTOR TRIPLE PRODUCT

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VECTOR TRIPLE PRODUCT

The vector triple product is given by:

$$\vec{p} = \vec{a} \times \left(\vec{b} \times \vec{c}\right) \tag{1}$$

Assuming that $\vec{b}$ and $\vec{c}$ are in the plane π, the vector $\left(\vec{b} \times \vec{c}\right)$ is perpendicular with respect to this plane, as shown in Figure 1. The vector $\vec{a}$ can be decomposed in a component contained in the plane π and a component perpendicular to it, being:

$$\vec{a} = \vec{a}_\pi + \vec{a}_n \tag{2}$$

According to the distributive property of the cross product it results:

$$\vec{p} = \vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{a}_\pi \times \left(\vec{b} \times \vec{c}\right) \tag{3}$$

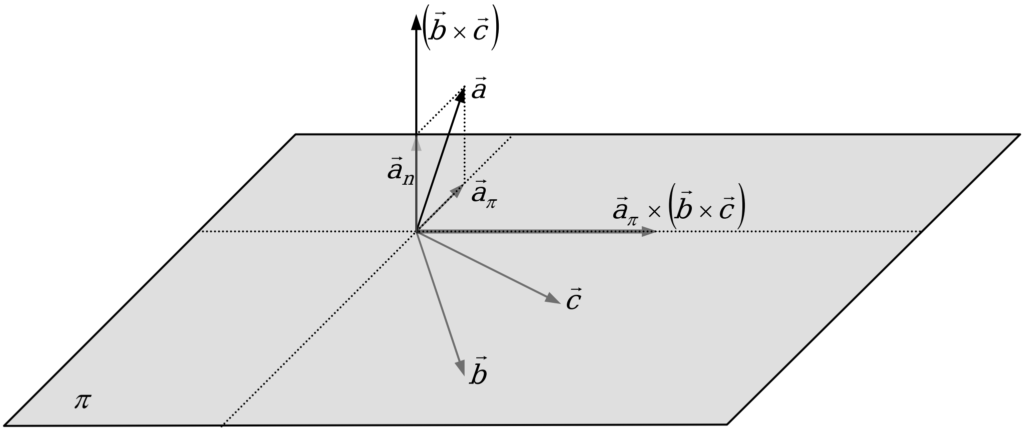

Figure 1. Vector triple product. Components of vectors contained in the plane π and perpendicular with respect to it.

Thus, the analysis can be carried out in the plane π, as shown in Figure 2. As the double product is contained in the plane π, it can be decomposed according to the directions of $\vec{b}$ and $\vec{c}$:

$$\vec{p} = \lambda_b \vec{u}_b + \lambda_c \vec{u}_c \tag{4}$$

Where $\vec{u}_b$ and $\vec{u}_c$ are the following unit vectors:

$$\vec{u}_b = \frac{\vec{b}}{b} \qquad \vec{u}_b = \frac{\vec{c}}{c} \tag{5}$$

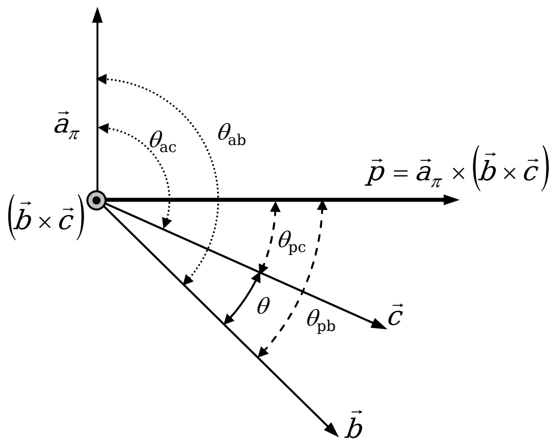

Figure 2. Components of vectors contained in the plane π definition of angles.

After multiplying scalarly Equation (4) by the unit vectors of Equation (5) it results:

$$\vec{p} \cdot \vec{u}_b = \lambda_b + \lambda_c\left(\vec{u}_c \cdot \vec{u}_b\right) \tag{6}$$

$$\vec{p} \cdot \vec{u}_c = \lambda_b\left(\vec{u}_b \cdot \vec{u}_c\right) + \lambda_c \tag{7}$$

According to the angles and vectors defined in Figure 2, Equations (6) and (7) can be written as

$$\left|\vec{p}\right| \cos\theta_{pb} = \lambda_b + \lambda_c \cos\theta \tag{8}$$

$$\left|\vec{p}\right| \cos\theta_{pc} = \lambda_b \cos\theta + \lambda_c \tag{9}$$

By multiplying Equation (9) by $\cos\theta$ and substracting it from Equation (8):

$$\left|\vec{p}\right|\left(\cos\theta_{pb} - \cos\theta_{pc}\cos\theta\right) = \lambda_b \sin^2\theta \tag{10}$$

Taking into account that $\theta_{pb} = \left(\theta_{pc} + \theta\right)$ and $\left|\vec{p}\right| = a_\pi bc \sin\theta$, Equation (10) is:

$$\lambda_b = -a_\pi bc \sin\theta_{pc} \tag{11}$$

According to Figure 2, $\theta_{pc} = \left(\theta_{ac} - \frac{\pi}{2}\right)$. Replacing int Equation (11) it results that:

$$\lambda_b = b a_\pi c \cos\theta_{ac} \tag{12}$$

Otherwise, according to the distributive property of the scalar product:

$$\vec{a} \cdot \vec{c} = \left(\vec{a}_\pi + \vec{a}_n\right) \cdot \vec{c} = \vec{a}_\pi \cdot \vec{c} = a_\pi c \cos\theta_{ac} \tag{13}$$

Combining Equations (12) and (13):

$$\lambda_b = b\left(\vec{a} \cdot \vec{c}\right) \tag{14}$$

In order to obtain λ_c, by multiplying Equation (8) by $\cos\theta$ and substracting it from Equation (9):

$$|\vec{p}|(\cos\theta_{pc} - \cos\theta_{pb}\cos\theta) = \lambda_c \sin^2\theta \tag{15}$$

Taking into account that $\theta_{pc} = (\theta_{pb} - \theta)$, Equation (15) becomes:

$$\lambda_c = a_\pi bc \sin\theta_{pb} \tag{16}$$

According to Figure 2, $\theta_{pb} = \left(\theta_{ab} - \frac{\pi}{2}\right)$. Replacing in Equation (16) it results:

$$\lambda_c = -ca_\pi b \cos\theta_{ab} \tag{17}$$

In a similar manner than in Equation (13) it results that:

$$\vec{a} \cdot \vec{b} = a_\pi b \cos\theta_{ab} \tag{18}$$

By comparing Equations (17) and (18):

$$\lambda_c = -c(\vec{a} \cdot \vec{b}) \tag{19}$$

Combining Equations (4), (5), (14) and (20) it is obtained:

$$\vec{p} = \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \tag{20}$$

The *e*-*δ* IDENTITY

Being $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ the unit vectors of a orthonormal basis, according to the definition of scalar and vector product it results:

$$\begin{aligned} \vec{e}_i \cdot \vec{e}_j &= \delta_{ij} \\ \vec{e}_i \times \vec{e}_j &= e_{ijk}\vec{e}_k \end{aligned} \tag{21}$$

Where δ_{ij} is the Kronecker delta and e_{ijk} the e-permutation symbol. The vector triple product of the unit vectos according to Equation (20) is:

$$\vec{e}_n \times (\vec{e}_i \times \vec{e}_j) = \vec{e}_i(\vec{e}_n \cdot \vec{e}_j) - \vec{e}_j(\vec{e}_n \cdot \vec{e}_i) \tag{22}$$

According to Equations (21), Equation (22) becomes:

$$\begin{aligned} e_{ijk}(\vec{e}_n \times \vec{e}_k) &= \vec{e}_i\delta_{nj} - \vec{e}_j\delta_{ni} \\ e_{ijk}e_{nkp}\vec{e}_p &= \vec{e}_i\delta_{nj} - \vec{e}_j\delta_{ni} \end{aligned} \tag{23}$$

By multiplying scalarly Equation (23) by $\vec{e}_m$:

$$e_{ijk}e_{nkp}\delta_{mp} = \delta_{im}\delta_{nj} - \delta_{jm}\delta_{ni} \tag{24}$$

Operating in the left member of Equation (24) the e-δ identity is:

$$e_{ijk}e_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{25}$$