Viscoelasticity

References

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Newton's law of viscous deformation. Subject to a load, a liquid flows. The amount of deformation is time-dependent and unlimited. When the load is removed, the liquid does not recover its original shape. The liquid has no memory.

To characterize deformation of this kind, we subject the liquid to a state of shear stress τ, and then measure the shear strain γ as a function of time t. The record, $\gamma(t)$, allows us to calculate the strain rate $d\gamma / dt$. For some liquids, the experimental data may be fit to a linear relation:

$$\frac{d\gamma}{dt} = \frac{\tau}{\eta}.$$

This linear fit is known as Newton's law of viscous deformation. The fitting parameter, η, is known as the viscosity of the liquid. The viscosity has the dimension of time×stress, and is specific to each liquid; e.g., water (η ~ 10^{-3} sPa) and glass (η ~ 10^{12} sPa).

Arrhenius equation. For a given liquid, the viscosity decreases as the temperature increases. A liquid flows when molecules pass one another, a process that is thermally activated. Thus, the relation between the viscosity and the temperature is often fit to the Arrhenius equation

$$\eta = \eta_0 \exp\left(\frac{q}{kT}\right).$$

The pre-factor η_0 and the activation energy q are parameters used to fit experimental data. kT is the temperature in units of energy. If we would rather use the Kelvin (K) as the unit for temperature, the conversion factors between the Kelvin and other units of energy are $k = 1.38 \times 10^{-23}\,\mathrm{JK}^{-1} = 8.63 \times 10^{-5}\,\mathrm{eVK}^{-1}$. We may measure the viscosity of a liquid at several temperatures, and then plot the data on the Arrhenius plot, namely, $\log\eta$ vs. $1/kT$. The activation energy is the slope of the plot, and the pre-factor η_0 is the intercept.

Mechanisms of deformation. How a material deforms depends on the constitution of the material.

Viscous strain. Under a load, a material deforms indefinitely. Molecules change neighbors. Pictorial Representation: Dashpot. The experimental data is fit to

$$\frac{d\varepsilon}{dt} = \frac{\sigma}{H}$$

Elastic strain. Under a load, the solid deforms instantaneously. Upon load removal, the solid recovers its original shape instantaneously. Bond stretching in crystals. Chains straighten up in rubbers (cross-linked polymers). Molecules do not change neighbors. Pictorial Representation: Spring. The experimental data is fit to

$$\varepsilon = \frac{\sigma}{E}.$$

Creep test of a viscoelastic rod. A rod of a polymer is held at a constant temperature, stress free, for a long time, so that its length no longer changes with time. The rod is in a state of equilibrium. At time zero, we hang a weight to the rod, and record its elongation as a function of time. Thus, the stress in the rod is a step function of time, but the strain is an increasing function of time, with the following characteristics:

- *Unrelaxed strain,* ε_U. Instantaneous after we hang the weight to the rod, the rod elongates by a certain amount. The molecules are not in a state of thermodynamic equilibrium, and will move to attain a state of equilibrium.
- *Relaxed strain,* ε_R. After some time, the molecules reach a new state of equilibrium, and the strain no long changes with time. This behavior is different from a polymer melt or any liquid, which will flow indefinitely under a constant stress.
- *Relaxation time,* τ. The time needed for the rod to change from the old to the new state of equilibrium.

Sketch the function $\varepsilon(t)$, and mark the above quantities. For some materials, the experimental data may fit to a formula:

$$\varepsilon(t) = \varepsilon_U + (\varepsilon_R - \varepsilon_U)\left[1 - \exp\left(-\frac{t}{\tau}\right)\right].$$

A molecular picture may look like this. When a weight is put on, molecules change configurations by two processes. A fast process occurs right after the weight is put on. A slow process takes place over some time. Of course, molecules may change configuration by more than two processes, and the experimental record of $\varepsilon(t)$ may not fit to the above formula.

Linear viscoelasticity. The magnitude of the strain should increase with the weight that we hang to the rod. Say at time zero we load two identical rods, one with weight W, and the other with weight $2W$. At any given time, the elongation of the second rod is twice that of the first rod. This linearity holds when the weight is small. We fit this experimental observation by

$$\varepsilon(t) = D(t)\sigma,$$

where σ is the suddenly applied stress, $\varepsilon(t)$ is the strain as a function of time, and $D(t)$ is the compliance as a function of time.

We can reinterpret the features of the function $\varepsilon(t)$, and denote the unrelaxed compliance by D_U, and the relaxed compliance by D_R. We may fit the experimental record with a formula:

$$D(t) = D_U + (D_R - D_U)\left[1 - \exp\left(-\frac{t}{\tau}\right)\right].$$

We next try to interpret the experimental observations. Your basic instinct might tell you to look how molecules pass one another, and try to relate various parameters to all the moving and jamming of the molecules. This approach has been tried as long as the existence of molecules was first appreciated, and has been dubbed as multi-scale modeling in recent years. The approach, however, is time consuming, and often not very practical. If you suspend your basic instinct for the time being, you may learn something from simple spring-dashpot models.

Maxwell model. This model represents a material with a spring in series with a dashpot. Because the two elements, the spring and the dashpot, are subject to the same stress, the model is also known as an iso-stress model. The total strain is the sum of the elastic and the viscous strain, so that

$$\frac{d\varepsilon}{dt}=\frac{1}{E}\frac{d\sigma}{dt}+\frac{\sigma}{H}.$$

This is an ODE. For a prescribed history of stress, the ODE can determine the history of strains

For example, in the creep test, we prescribe the history of stress as a step function:

$$\sigma(t)=\begin{cases}0, & t<0\\ \sigma, & t\geq 0\end{cases}$$

The history of strain will be

$$\varepsilon(t)=\begin{cases}0, & t<0\\ \dfrac{\sigma}{E}+\dfrac{\sigma}{H}t, & t\geq 0\end{cases}$$

We may also write the expression as

$$\varepsilon(t)=\frac{\sigma}{E}\left(1+\frac{t}{\tau}\right),$$

where

$$\tau=H/E$$

is a time scale. The model gives the creep compliance:

$$D(t)=\frac{1}{E}\left(1+\frac{t}{\tau}\right).$$

The Maxwell model gives an unrelaxed strain, but not a relaxed strain. The dashpot is unconstrained: the rod will flow indefinitely under the weight, and will never attain an equilibrium state. The long-time behavior is liquid-like.

Kelvin model. This model represents a material with a spring in parallel with a dashpot. Because the two elements are subject to the same strain, the model is also known as an iso-strain model. The total stress σ is the sum of the stress in the spring and the stress in the dashpot:

$$\sigma=E\varepsilon+H\frac{d\varepsilon}{dt}.$$

This equation relates a history of stress to a history of strain.

In the creep test, the stress is zero when $t<0$, and is held at a constant level σ when $t>0$. We now try to solve for the history of strain, $\varepsilon(t)$. The above equation becomes an inhomogenous ODE with constant coefficients. The general solution is

$$\varepsilon(t)=\frac{\sigma}{E}+A\exp\left(-\frac{t}{\tau}\right),$$

where A is a constant of integration, and

$$\tau=H/E$$

is the relaxation time. At $t=0$, the dashpot allows no strain, $\varepsilon(0)=0$. This initial condition determines the constant A, so that the history of strain is

$$\varepsilon(t)=\frac{\sigma}{E}\left[1-\exp\left(-\frac{t}{\tau}\right)\right].$$

The strain is linear in the stress, and the creep compliance is

$$D(t)=\frac{1}{E}\left[1-\exp\left(-\frac{t}{\tau}\right)\right].$$

The Kelvin model gives a relaxed strain, but not an unrelaxed strain. In parallel, the spring cannot elongate instantaneously, and dashpot stops elongating eventually. The short-time behavior is unrealistic, and the long-time behavior is solid like.

Zener model. Neither the Maxwell nor the Kelvin model is sufficient to describe the representative experimental data. To describe the above observations, we need a combination of the Maxwell and the Kelvin model. In the Zener model, a spring of modulus E_1 and a dashpot of viscosity H are in parallel, and form a Kelvin unit. The unit is then in series with another spring of modulus E_0. Let ε_1 be the strain due to the Kelvin unit, so that

$$\sigma = E_1\varepsilon_1 + H\frac{d\varepsilon_1}{dt} ,$$

giving

$$\varepsilon_1 = \frac{1}{E_1}\left[1-\exp\left(-\frac{t}{\tau}\right)\right]\sigma ,$$

where the relaxation time is $\tau = H / E_1$.

The total strain $\varepsilon(t)$ is the sum of the strain due to the Kelvin unit and the strain due to the spring in series:

$$\varepsilon(t)=\frac{\sigma}{E_0}+\frac{1}{E_1}\left[1-\exp\left(-\frac{t}{\tau}\right)\right]\sigma ,$$

The creep compliance is

$$D(t)=\frac{1}{E_0}+\frac{1}{E_1}\left[1-\exp\left(-\frac{t}{\tau}\right)\right].$$

Thus, the unrelaxed compliance is

$$D_U = \frac{1}{E_0}.$$

The relaxed compliance is

$$D_R = \frac{1}{E_0}+\frac{1}{E_1}.$$

A polymer may have several mechanisms of relaxation, each with a distinct relaxation time. We can model the polymer by using a series of Kelvin units, together with a spring. Each Kelvin unit represents a mechanism of relaxation, operating at a different time scale. The creep compliance will take the form

$$D(t)=\frac{1}{E_0}+\frac{1}{E_1}\left[1-\exp\left(-\frac{t}{\tau_1}\right)\right]+\frac{1}{E_2}\left[1-\exp\left(-\frac{t}{\tau_2}\right)\right]+\ldots$$

Boltzmann superposition principle. Now we load a rod with a known history of stress, $\sigma(t)$. What will be the history of strain, $\varepsilon(t)$? Of course, for each given history of stress, we can always measure the history of strain experimentally, but that is a lot of measurements. We now describe a linear superposition principle.

Say we have conducted a creep test and measured the creep compliance, $D(t)$. The creep compliance is zero when $t < 0$, and increases when $t > 0$. Remember, $D(t)$ is measured by subject the solid to a step stress.

We next approximate a history of stress by a series of steps:

- At time $t = u_0$, the stress steps from zero to σ_0.
- At time $t = u_1$, the stress makes an additional step $\sigma_1 - \sigma_0$
- At time $t = u_2$, the stress makes yet another step $\sigma_2 - \sigma_1$

The total strain is the sum of the strains due to all the steps:

$$\varepsilon(t) = \sigma_0 D(t - u_0) + (\sigma_1 - \sigma_0) D(t - u_1) + (\sigma_2 - \sigma_1) D(t - u_2).$$

As an example, consider a rod subject to a stress at time zero. The stress is kept constant for a while and then removed at time t_1. The rod gradually contracts to its original length. The stress is zero when $t < 0$, is held at a constant level σ between $0 < t < u_1$, and is dropped to zero again at $t > u_1$. The history of strain will be

$$\varepsilon(t) = \sigma D(t) - \sigma D(t - u_1).$$

Sketch this history. At time zero, the strain steps up by $D_U \sigma$, and then increases with the time. At time u_1, the strain steps down by $D_U \sigma$, and then gradually decays to zero.

A rod is held at a constant temperature, stress free, for a long time, and no longer changes its length. This length is used as the reference to calculate the strain. We then apply a history of stress $\sigma(t)$. The history of strain is given by a superposition:

$$\varepsilon(t) = \sum D(t - u_i)[\sigma(u_{i+1}) - \sigma(u_i)],$$

or,

$$\varepsilon(t) = \int_{-\infty}^{t} D(t - u) \frac{d\sigma(u)}{du} du.$$

In this equation, t is the time at which the strain is measured, and u is the variable of integration (i.e., a dummy variable). The lower limit of the integral is -∞ because the complete stress history prior to t contributes to the observed strain. The upper limit of the integral is t, because the stress applied after t had no effect on the strain measured at time t.

Relaxation test. At time zero, apply a constant strain to a rod, and then record the stress in the rod as a function of time. The stress decreases as a function of time. Main characteristics include

- σ_U the unrelaxed stress is obtained instantaneously after the application of the step-strain.
- σ_R the relaxed stress is obtained after a long time of the application of the step-strain.
- β is the characteristic time for stress relaxation. For the same material, the characteristic time for stress relaxation test is different from the characteristic time for the creep test.

If the applied strain is sufficiently small, the stress is linear in the strain, namely,

$$\sigma(t) = E(t)\varepsilon$$

The stress relaxation modulus is a function of time, $E(t)$. A suitable curve to fit the experimental data may be

$$E(t) = E_R + (E_U - E_R)\exp\left(-\frac{t}{\beta}\right).$$

.

From physical considerations, the compliance and the modulus at the two limit states are related:

$$E_U = 1/D_U, E_R = 1/D_R.$$

The relaxation time determined inn the relaxation test, β, is different from the relaxation time determined from the creep test, τ.

Relaxation test analyzed using the Zener model. The applied strain is the sum of the strain due to the Kelvin unit and the strain due to the spring in series:

$$\varepsilon = \varepsilon_1 + \frac{\sigma}{E_0}$$

For the Kelvin unit, the total stress is the sum of the stress due to the sping and that due to the dashpot:

$$\sigma = E_1\varepsilon_1 + H\frac{d\varepsilon_1}{dt}.$$

Eliminating ε_1 from the above two equations, we obtain that

$$H\frac{d\sigma}{dt} + (E_1 + E_0)\sigma = E_1E_0\varepsilon.$$

Note the initial condition $\varepsilon_1(0) = 0$ and $\sigma(0) = E_0\varepsilon$. The solution to this ODE is

$$\sigma = E_R\varepsilon + (E_U - E_R)\exp\left(-\frac{t}{\beta}\right)\varepsilon,$$

where

$$E_R = \frac{E_1E_0}{E_1 + E_0}, \quad E_U = E_0, \quad \beta = \frac{H}{E_1 + E_0}$$

The Zener model relates the relaxation times of the two types of tests:

$$\tau / \beta = E_U / E_R.$$

Cyclic test. Program a machine to prescribe to a rod a history of cyclic strain:

$$\varepsilon(t) = \varepsilon_0 \sin \omega t,$$

where ε_0 is the amplitude, and ω the frequency. What will be the history of stress $\sigma(t)$? The answer depends on material.

For a *purely elastic solid*, the stress is proportional to the strain at all time, $\sigma(t) = E\varepsilon(t)$. When the machine prescribes a sinusoidal strain $\varepsilon(t) = \varepsilon_0 \sin \omega t$ to the rod, the stress in the rod is

$$\sigma(t) = E\varepsilon_0 \sin \omega t.$$

The stress is *in phase* with the strain.

For a *purely viscous solid*, the stress is proportional to the strain rate, $\sigma(t) = Hd\varepsilon / dt$. When the machine is programmed to apply a sinusoidal strain $\varepsilon(t) = \varepsilon_0 \sin \omega t$ to the rod, the stress in the rod is $\sigma(t) = H\varepsilon_0\omega \cos \omega t$. This expression can be rewritten as

$$\sigma(t) = H\varepsilon_0\omega \sin\left(\omega t + \frac{\pi}{2}\right)$$

The stress is out of phase with the strain by a shift $\pi / 2$.

Now consider a *viscoelastic solid*. When the machine dictates the solid to strain according to $\varepsilon(t) = \varepsilon_0 \sin \omega t$, the stress as a function of time looks complicated in the first few cycles. Afterwards, the stress becomes cyclic, with the same frequency ω, but with a phase shift. This cyclic stress is written as

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta).$$

The phase shift δ is between 0 and $\pi / 2$.

Storage and loss. When a linearly viscoelastic material is subject to the cyclic strain,

$$\varepsilon(t) = \varepsilon_0 \sin \omega t ,$$

the history of stress is $\sigma(t) = \sigma_0 \sin(\omega t + \delta)$, namely,

$$\sigma(t) = \sigma_0 \cos\delta \sin\omega t + \sigma_0 \sin\delta \cos\omega t$$

The stress consists of two components: one in-phase with the strain, and the other out-of-phase with the strain.

Consider a rod of *L* and cross-sectional area *A*. The rod elongates as

$$\Delta(t) = L\varepsilon_0 \sin\omega t.$$

The machine applies to the rod a force

$$F(t) = A\sigma_0(\sin\omega t \cos\delta + \cos\omega t \sin\delta).$$

During a time interval dt, the solid elongates by $d\Delta$, and the machine does work $Fd\Delta$ to the solid. In a full cycle the machine does work:

$$\int_0^{2\pi/\omega} F \frac{d\Delta}{dt} dt = \pi AL\sigma_0\varepsilon_0 \sin\delta.$$

In a full cycle, the elastic energy stored in the rod repeats its value. Consequently, the net work done by the machine in a full cycle must all become heat. The above expression is the energy dissipated by the rod per cycle. The in-phase stress and strain results in the elastic energy which is completely recoverable. The $\pi / 2$ out-of-phase stress and strain results in the dissipated energy.

We will call

$$E' = \frac{\sigma_0 \cos\delta}{\varepsilon_0}$$

the storage modulus, and

$$E'' = \frac{\sigma_0 \sin\delta}{\varepsilon_0}$$

the loss modulus. The loss tangent is defined as

$$\tan\delta = \frac{E''}{E'}.$$

Complex numbers. To ease calculation, we will use complex numbers. Let us first recall a few facts in the theory of complex numbers. Let $i = \sqrt{-1}$, and *a* and *b* be two real numbers. Write a complex number *z* as

$$z = a + ib ,$$

where *a* is the real part of *z*, and *b* the imaginary part of *z*.

The complex number *z* can also be represented by a point in a plane, with *a* and *b* as the two coordinates of the point. Consider the vector from the origin of the plane to the point (a, b). Let r be the magnitude of the vector, and θ be the slope of the vector. Thus,

$$a = r\cos\theta, \qquad b = r\sin\theta.$$

Recall Euler's identity

$$\exp(i\theta) = \cos\theta + i\sin\theta .$$

Thus,

$$z = r\exp(i\theta).$$

We call r the magnitude of z, and θ the phase angle of z.

Recall identities

$$\exp(u)\exp(v) = \exp(u+v),$$

$$\frac{d\exp(w)}{dw} = \exp(w).$$

Complex modulus. Write strain as

$$\varepsilon(t) = \varepsilon^* \exp(i\omega t),$$

where ε^* is in general a complex amplitude independent of time. Similarly, write stress as

$$\sigma(t) = \sigma^* \exp(i\omega t),$$

where σ^* is in general a complex amplitude independent of time.

As a material model, we assume that the complex stress is linear in the complex strain,

$$\sigma^* = E^* \varepsilon^*,$$

where E^* be the *complex modulus*.

The complex modulus consists of a real and an imaginary part:

$$E^* = E' + iE''.$$

We will see that they correspond to the storage modulus and the loss modulus. Let $\hat{E}$ be the magnitude of the complex modulus, and δ be the phase angle of the complex modulus, so that

$$E^* = \hat{E}\exp(i\delta).$$

Of course, both $\hat{E}$ and δ are real-valued. The storage modulus and the loss modulus are given by

$$E' = \hat{E}\cos\delta, \quad E'' = \hat{E}\sin\delta .$$

The history of stress relates to the history of the strain as

$$\sigma(t) = \hat{E}\exp(i\delta)\varepsilon(t).$$

This equation indicates that the phase of the stress shifts by δ relative to the phase of the strain.

How do complex numbers simplify algebra? As we will see from the following examples, we no longer need to deal with two functions, sin and cos. Every term of an equation has the function exp, so that exp can be factored out.

At the end of the calculation, we will just take the imaginary part to recover the real-valued strain and stress. For example, let $\varepsilon^* = \varepsilon_0$, which is real-valued. The imaginary part of $\varepsilon(t) = \varepsilon^* \exp(i\omega t)$ recovers the real-valued strain $\varepsilon(t) = \varepsilon_0 \sin\omega t$. The stress becomes $\sigma(t) = \hat{E}\varepsilon_0 \exp(i\omega t + i\delta)$. Taking the imaginary part, we recover the real-valued stress $\sigma(t) = \hat{E}\sigma_0 \sin(\omega t + \delta)$.

Cyclic test analyzed using the Zener model. Recall the Zener model:

$$\varepsilon = \varepsilon_1 + \frac{\sigma}{E_0}, \quad \sigma = E_1\varepsilon_1 + H\frac{d\varepsilon_1}{dt} .$$

The model relates the three time-dependent functions: $\varepsilon(t), \sigma(t), \varepsilon_1(t)$. In a cyclic test, the machine is programmed to apply a sinusoidal strain, $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$. We wish to solve for $\sigma(t)$ and $\varepsilon_1(t)$ from the Zener model. The solution depends on the initial condition. After some time, however, both $\sigma(t)$ and $\varepsilon_1(t)$ will become sinusoidal, with some phase shift relative to the applied history of strain: the effect of the initial condition will die out. It is this steady state that interests us. We can of course solve this problem by using real-valued functions, but complex-valued functions will simplify the algebra.

We look for the steady-state solution of the form:

$$\varepsilon(t) = \varepsilon^* \exp(i\omega t)$$

$$\sigma(t) = \sigma^* \exp(i\omega t)$$

$$\varepsilon_1(t) = \varepsilon_1^* \exp(i\omega t)$$

In terms of the complex amplitudes, the model becomes

$$\varepsilon^* = \varepsilon_1^* + \frac{\sigma^*}{E_0}, \quad \sigma^* = (E_1 + iH\omega)\varepsilon_1^*.$$

Note that the ODE has become an algebraic equation. For a given input ε^*, the pair of algebraic equations determine ε_1^* and σ^*.

Eliminating ε_1^*, we obtain that

$$\sigma^* = \frac{E_0(E_1 + iH\omega)}{E_0 + E_1 + iH\omega}\varepsilon^*$$

Recall the definition of the complex modulus, $\sigma^* = E^*\varepsilon^*$. Thus,

$$E^* = \frac{E_0(E_1 + iH\omega)}{E_0 + E_1 + iH\omega}.$$

The various parameters in the Zener model relate to the parameters determined from a relaxation test:

$$E_R = \frac{E_1 E_0}{E_1 + E_0}, \quad E_U = E_0, \quad \beta = \frac{H}{E_1 + E_0}.$$

Thus,

$$E^* = E_U + \frac{E_R - E_U}{1 + i\omega\beta},$$

Taking the real and the imaginary part, we have

$$E'(\omega) = E_U + \frac{E_R - E_U}{1 + (\omega\beta)^2}, \quad E''(\omega) = \frac{(E_U - E_R)\omega\beta}{1 + (\omega\beta)^2}$$

Both the storage modulus and the loss modulus are functions of the loading frequency. Sketch the plot. The following remarks conform to our intuition about viscoelasticity.

- When the loading frequency is low, $\beta\omega \ll 1$, the storage modulus is the same as the relaxed modulus, $E'(0) = E_R$, and the loss modulus vanishes, $E''(0) = 0$.
- When the loading frequency is high, $\beta\omega >> 1$, the storage modulus is the same as the unrelaxed modulus, $E'(\infty) = E_U$, and the loss modulus vanishes $E''(\infty) = 0$.
- At some intermediate frequency, $\beta\omega = 1$, the storage modulus increases rapidly, and the loss modulus peaks.

A plane wave in a viscoelastic solid. Viscoelasticity introduces a time scale into the problem, so that the waves must be dispersive. Instead of looking for waves of an arbitrary profile, we look for waves with a sinusoidal profile. To illustrate essential features, we will consider waves propagating in a rod.

First let us list the governing equations on the basis of the three ingredients of solid mechanics. The balance of momentum requires that

$$\frac{\partial \sigma(x,t)}{\partial x} = \rho \frac{\partial^2 u(x,t)}{\partial t^2}.$$

The strain-displacement equation is

$$\varepsilon(x,t) = \frac{\partial u(x,t)}{\partial x}.$$

These two ingredients are independent of materials. The third ingredient, the material model, now invokes viscoelasticity.

We now look for such a field that all material particles vibrate at the same frequency ω. We will use the complex numbers:

$$u(x,t) = u^*(x,\omega)\exp(i\omega t)$$

$$\varepsilon(x,t) = \varepsilon^*(x,\omega)\exp(i\omega t)$$

$$\sigma(x,t) = \sigma^*(x,\omega)\exp(i\omega t)$$

Using the complex amplitudes, the three ingredients of solid mechanics are written as follows. The balance of momentum now requires that

$$\frac{\partial \sigma^*(x,\omega)}{\partial x} = -\rho\omega^2 u^*(x,\omega).$$

The strain-displacement equation now becomes

$$\varepsilon^*(x,\omega) = \frac{\partial u^*(x,\omega)}{\partial x}.$$

The materials model is

$$\sigma^*(x,\omega) = E^*(\omega)\varepsilon^*(x,\omega).$$

A combination of the three ingredients gives that

$$E^* \frac{\partial^2 u^*(x,\omega)}{\partial x^2} + \rho\omega^2 u^*(x,\omega) = 0.$$

The solution is

$$u^*(x,\omega) = A\exp\left(-i\sqrt{\frac{\rho}{E^*}}\omega x\right) + B\exp\left(i\sqrt{\frac{\rho}{E^*}}\omega x\right).$$

A wave propagating in the positive direction of *x* takes the form

$$u(x,t) = A\exp\left(-i\sqrt{\frac{\rho}{E^*}}\omega x + i\omega t\right).$$

Write

$$E^* = \hat{E}\exp(i\delta).$$

The above solution becomes

$$u(x,t) = A\exp\left(-\sqrt{\frac{\rho}{\hat{E}}}\omega \sin\left(\frac{\delta}{2}\right)x\right)\exp\left(-i\sqrt{\frac{\rho}{\hat{E}}}\omega \cos\left(\frac{\delta}{2}\right)x + i\omega t\right)$$

The wave propagates at the velocity

$$c(\omega)=\frac{1}{\cos(\delta/2)}\sqrt{\frac{\hat{E}}{\rho}}\ .$$

The decay length is

$$l(\omega)=\frac{1}{\omega\sin(\delta/2)}\sqrt{\frac{\hat{E}}{\rho}}\ .$$

As indicated, both quantities are functions of the frequency. The two functions are determined when we know $\hat{E}(\omega)$ and $\delta(\omega)$.

At a low frequency, $\omega\beta << 1$, the viscoelastic solid behaves as an elastic solid with the relaxed modulus, so that $\hat{E} = E_R$ and $\delta = 0$. The wave speed is

$$c(0)=\sqrt{\frac{E_R}{\rho}}\ .$$

The decay length approaches infinity:

$$l(0)\to\infty\ .$$

At a high frequency, $\omega\beta >> 1$, the wave speed is

$$c(\infty)=\sqrt{\frac{E_U}{\rho}}\ .$$

The decay length is

$$l(\infty)=\frac{2\beta}{1-\frac{E_R}{E_U}}\sqrt{\frac{E_U}{\rho}}\ .$$

This concludes our brief introduction to viscoelasticity. The book by Ferry contains descriptions of experiments and data. The book by Christensen contains many analytical techniques of solving boundary-value problems. AQAQUS can be used to solve boundary-value problems.